Introduction to the Worldline Formalism

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Abstract

The following article introduces the so called Worldline Formalism in QFT and illustrates its applications to various problems historically handled by means of 2nd quantization and Feynman diagrams.

Contents

1 Conventions

We use natural units $\hbar = c = 1$. We work in D spacetime dimensions and use the signature $(+, -, .., -)$, denoting D-vectors $\mathbf{x}$ with **bold** symbols, their spatial part by $\vec{x}$. We denote by $g(\mathbf{x}, \mathbf{y}) = x^0 y^0 - \vec{x} \cdot \vec{y}$ the Minkowski-scalar-product of the D-vectors $\mathbf{x}, \mathbf{y}$. Greek indices run (unless stated otherwise) from 0 to $(D-1)$, latin ones only from 1 to $(D-1)$. Components of linear functionals are written down, components of vectors up. Operators are symbolized with a hat "$\hat{}$".

For two-component Grassmann variables $(\vartheta^1, \vartheta^2) =: \boldsymbol{\vartheta}$ we shall write $\boldsymbol{\vartheta}_k \boldsymbol{\vartheta}_l := \vartheta_k^1 \vartheta_l^1 + \vartheta_k^2 \vartheta_l^2$.

We use the isometric Fourier-transform.

2 Introduction

The path integral formalism developed by Feynman[4] in the first half of the 20th century, has come to be a successful complement to the non-relativistic quantum mechanical formalism developed by Schrödinger and Heisenberg. Furthermore, its apparent simplicity and similarity to classical mechanical variational methods, makes its generalization to functional integrals over fields, a promising alternative to the 2nd quantization description used in quantum field theory and in particular the calculation of N-point correlators[1].

In the following years, a new approach to the calculation of generating functionals and Green's functions in field theory had been developed mainly by Fock, Feynman[11] and Schwinger[20]. This so called *worldline formalism*, can be used to solve a theory using solely one-parameter path integrals.

Though simple problems such as the ones handled below, can generally be solved much easier using 2nd quantization methods and Wick contractions, the worldline formalism shows its real strength in more complex situations. In particular, *worldline numerics* has lately enjoyed a wide span of applications ranging from the Casimir effect[25, 26] to pair production in inhomogeneous fields[27, 28]. Furthermore, the intuitive interpretation of worldline integrals, allows for a completely different insight into the whole theory, just as path integrals do in quantum mechanics.

Finally, the worldline formalism shows great analogy to similar results in string theory, where string scattering amplitudes reduce to one-parameter path integral expressions. In fact, as string theory reduces to quantum field theory in the infinite tension limit, string scattering amplitudes reduce to quantum field theoretic ones, thus *inspiring* the development of a worldline formalism eventually without reference to string theory[5, 14, 24].

The following, somewhat quick and dirty, article shall introduce this formalism and illustrate its applications to various problems historically handled by means of 2nd quantization and Feynman diagrams. It is mainly based on papers [5],[11],[14],[15] and [19].

Basic knowledge of one-parameter path-integrals[1, 2, 23] and Grassmann variables[1, 8, 12] is assumed.

2.1 Path integral formulation of the KG Green's function

In his paper "Mathematical Formulation of the Quantum Theory of Electromagnetic Interaction"[11], 1950, Feynman introduced a path-integral description of spin-0 particles, *for its own interest as an alternative to the formulation of second quantization*. In it, he expressed the Green's function of the Klein-Gordon operator $\Box + m^2$, as a path integral over spacetime paths along a 5th parameter. We shall in the following briefly illustrate his ideas, mainly as a motivation for what follows thereafter. Consider the KG equation

$$\left(\boxdot + \sigma + m^2\right) \varphi(\mathbf{x}) = 0 \quad , \quad \boxdot := (\partial^\mu - ieA^\mu)(\partial_\mu - ieA_\mu) \tag{2.1}$$

for a spinless complex field φ of charge e in a background gauge field $\mathbf{A}$ with source $\sigma(\mathbf{x})$. Consider now the equation[2]

$$i\partial_u \zeta(\mathbf{x}, u) = \hat{H} \zeta(\mathbf{x}, u) \quad , \quad \hat{H} := \boxdot + \sigma \tag{2.2}$$

[1]See for example Greiner[13].
[2]Here, u is an additional parameter to the parameter $\mathbf{x}$ in $\varphi(\mathbf{x})$.

for the field $\zeta(\mathbf{x},u)$. Since $\hat{H}$ does not depend on u, solutions of (2.2) can be written as combinations of the fundamental solutions

$$\zeta(\mathbf{x},u) = e^{-iEu}\varphi(\mathbf{x}), \tag{2.3}$$

whereas $\hat{H}\varphi \overset{!}{=} E\varphi$. In particular, φ solves (2.1) iff $E=-m^2$. Thus, for any solution $\zeta(\mathbf{x},u)$ of (2.2),

$$\varphi(\mathbf{x}) := \int du\; e^{-im^2u}\zeta(\mathbf{x},u) \tag{2.4}$$

solves (2.1). Now suppose $K(\mathbf{x},u;\mathbf{x}',0)$ solves (2.2) and satisfies[3]

$$K(\mathbf{x},0;\mathbf{x}',0) = \delta(\mathbf{x}-\mathbf{x}') \quad , \quad K(\mathbf{x},u;\mathbf{x}',0) = 0 \text{ for } u<0. \tag{2.5}$$

Then

$$G(\mathbf{x},\mathbf{x}') := i\int e^{-im^2u}K(\mathbf{x},u;\mathbf{x}',0)\; du \tag{2.6}$$

satisfies

$$\begin{aligned}
(\boxdot_{\mathbf{x}} + \sigma(\mathbf{x}) + m^2)G(\mathbf{x},\mathbf{x}') &= i\int_0^\infty e^{-im^2u}\underbrace{\hat{H}_{\mathbf{x}}\;K(\mathbf{x},u;\mathbf{x}',0)\; du}_{i\partial_u K(\mathbf{x},u;\mathbf{x}',0)} + m^2 G(\mathbf{x},\mathbf{x}') \\
&= -K(\mathbf{x},u;,\mathbf{x}',0)e^{-im^2u}\big|_{u=0}^{\infty} + \int_0^\infty K(\mathbf{x},u;\mathbf{x}',0)\partial_u e^{-im^2u} + m^2G(\mathbf{x},\mathbf{x}') \\
&= \underbrace{K(\mathbf{x},0;\mathbf{x}',0)}_{\delta(\mathbf{x}-\mathbf{x}')} - m^2\,\underbrace{i\int_0^\infty K(\mathbf{x},u;\mathbf{x}',0)e^{-im^2u}\; du}_{G(\mathbf{x},\mathbf{x}')} + m^2G(\mathbf{x},\mathbf{x}') \\
&= \delta(\mathbf{x}-\mathbf{x}'),
\end{aligned} \tag{2.7}$$

that is, G is a Green's function[4] for the Klein-Gordon operator $(\boxdot + \sigma + m^2)$. From the theory of path integrals in non-relativistic quantum mechanics, we know that the transition element for the *Schrödinger equation* (2.2) is given by[5]

$$K(\mathbf{x},u;\mathbf{x}',0) = \langle\mathbf{x}|\, e^{-iu\boxdot}\, |\mathbf{x}'\rangle = \int_{(0,\mathbf{x}')}^{(u,\mathbf{x})} D\mathbf{y}\;\exp\left[i\int_0^u \mathcal{L}(\mathbf{y},\dot{\mathbf{y}})\; d\tau\right] \quad , \quad u\geq 0, \tag{2.8}$$

with $\mathcal{L}$ as the Lagrangian corresponding to the *Hamiltonian*

$$\hat{H} = -(p^\mu - eA^\mu)(p_\mu - eA_\mu) + \sigma = -\hat{p}^\mu p_\mu - e^2A^\mu A_\mu + e\{A^\mu,\hat{p}_\mu\} + \sigma. \tag{2.9}$$

As described in appendix A.3, we obtain[6]

$$\mathcal{L}(\mathbf{y},\dot{\mathbf{y}}) = -\frac{1}{4}\dot{y}^\mu\dot{y}_\mu + e\dot{y}^\mu A_\mu(\mathbf{y}) - \sigma(\mathbf{y}) \quad . \tag{2.10}$$

[3] Thus, $K(\mathbf{x},u;,\mathbf{x}',0)$ represents the transition amplitude of a particle at point $\mathbf{x}'$ to $\mathbf{x}$ from time 0 to time $u>0$. Consequently, for any solution $\zeta(\mathbf{x},u)$ of (2.2),

$$\zeta(\mathbf{x},u) = \int d\mathbf{x}'\; K(\mathbf{x},u;\mathbf{x}',0)\zeta(\mathbf{x}',0)$$

holds for $u\geq 0$.

[4] For the domain $\mathbb{R}^4$, vanishing with $\|\mathbf{x}-\mathbf{x}'\|\to\infty$.

[5] The exact phase of the measure $D\mathbf{y}$ differs by the one for Hamiltonians with positive-definite mass-matrices $\mathbb{M}$.

[6] In his paper [11], Feynman used the operator $\frac{1}{2}\Box$ instead of $\Box$ for defining (2.2), thus obtaining (for a free particle) the path-integrant $e^{-i\int g(\dot{\mathbf{y}},\dot{\mathbf{y}})/2\; d\tau}$ instead. Consequently, the Green's function $G(\mathbf{x},\mathbf{x}')$ was given by the $e^{-i\frac{m^2}{2}u}$ mode of $K(\mathbf{x},u;\mathbf{x}',0)$.

The final expression for the Green's function thus becomes

$$\boxed{G(\mathbf{x},\mathbf{x}') := i\int_0^\infty du\; e^{-im^2u}\int_{(0,\mathbf{x}')}^{(u,\mathbf{x})} D\mathbf{y}\;\exp\left[-i\int_0^u \left[\frac{1}{4}\dot{y}^\mu\dot{y}_\mu - e\dot{y}^\mu A_\mu + \sigma\right]\, d\tau\right].} \tag{2.11}$$

3 Worldline formalism for Klein-Gordon fields

3.1 The effective action

We consider the generating functional

$$Z[\underbrace{J,J^*}_{=:\mathbf{J}}] := \int D\varphi\;\exp\left[i\int d^4\mathbf{x}\;\mathcal{L}^{\mathrm{KG}} + i\int d^4\mathbf{x}\;(J^*\varphi + \varphi^* J)\right] \tag{3.1}$$

for the complex Klein-Gordon field with background gauge field $\mathbf{A}$:

$$\mathcal{L}^{\mathrm{KG}}(\varphi,\varphi^*,\partial\varphi,\partial\varphi^*) = \frac{g^{\mu\nu}}{2}(D^*_\mu\varphi^*)(D_\nu\varphi) - \frac{m^2}{2}\varphi^*\varphi, \tag{3.2}$$

$$D_\mu := (\partial_{x^\mu} - iA_\mu) \quad , \quad D^*_\mu := (\partial_{x^\mu} + iA_\mu).$$

Performing a Wick-rotation to imaginary times $\widetilde{x}^0 := ix^0$, we obtain the representation

$$Z[\mathbf{J}] = \int D\varphi\;\exp\Bigg[\underbrace{\int d^4\widetilde{\mathbf{x}}\;\mathcal{L}_E^{\mathrm{KG}} + \int d^4\widetilde{\mathbf{x}}\;(J^*\varphi + \varphi^* J)}_{=:S_E^{\mathbf{J}}[\varphi,\varphi^*]}\Bigg] \tag{3.3}$$

with

$$\mathcal{L}_E^{\mathrm{KG}}(\varphi,\varphi^*,\widetilde{\partial}\varphi,\widetilde{\partial}\varphi^*) := -\frac{\delta^{\mu\nu}}{2}(\widetilde{D}^*_\mu\varphi^*)(\widetilde{D}_\nu\varphi) - \frac{m^2}{2}\varphi^*\varphi \tag{3.4}$$

$$\widetilde{D}_\mu := (\partial_{\widetilde{x}^\mu} - i\widetilde{A}_\mu) \quad , \quad \widetilde{D}^*_\mu := (\partial_{\widetilde{x}^\mu} + i\widetilde{A}_\mu) \quad , \quad \widetilde{\mathbf{A}} := (-iA_0, \vec{A})$$

as the Euclidean version of (3.2). Let $\boldsymbol{\varphi}_c := (\varphi_c, \varphi_c^*)$ extremize the action $S_E^{\mathbf{J}}$ and for any field $\boldsymbol{\varphi} := (\varphi,\varphi^*)$ set $\boldsymbol{\varphi}' := \boldsymbol{\varphi} - \boldsymbol{\varphi}_c$. Then, up to 2nd order in $\boldsymbol{\varphi}'$ (*1-loop accuracy*), we may write[7]

$$S_E^{\mathbf{J}}[\boldsymbol{\varphi}] = S_E^{\mathbf{J}}[\boldsymbol{\varphi}_c] + \underbrace{\left.\frac{dS_E^{\mathbf{J}}}{d\boldsymbol{\varphi}}\right|_c}_{0}\boldsymbol{\varphi}' + \frac{1}{2!}\left.\frac{d^2S_E^{\mathbf{J}}}{d\boldsymbol{\varphi}^2}\right|_c(\boldsymbol{\varphi}',\boldsymbol{\varphi}'), \tag{3.5}$$

[7] Note that this expansion (3.5) is exact, as $\mathcal{L}$ is at most quadratic in the fields.

where $\big|_c$ means evaluation at $\boldsymbol{\varphi}_c$. Consequently, (3.3) takes the form

$$
\begin{aligned}
Z^{11}[\mathbf{J}] &= e^{S_E^J[\varphi_c]} \cdot \int D\boldsymbol{\varphi}' \ \exp\left[\frac{1}{2!}\frac{d^2 S_E^{\mathbf{J}}}{d\boldsymbol{\varphi}^2}\bigg|_c (\boldsymbol{\varphi}', \boldsymbol{\varphi}')\right] \\
&= e^{S_E^{\mathbf{J}}[\boldsymbol{\varphi}_c]} \cdot \int D\boldsymbol{\varphi}' \ \exp\left[-\frac{1}{2}\int d^4\widetilde{\mathbf{x}} \ \left[\delta^{\mu\nu}(\widetilde{D}_\mu^*\varphi^{*\prime})(\widetilde{D}_\nu\varphi') + m^2\varphi^{*\prime}\varphi'\right]\right] \\
&= e^{S_E^{\mathbf{J}}[\boldsymbol{\varphi}_c]} \cdot \int D\boldsymbol{\varphi}' \ \exp\left[-\frac{1}{2}\int d^4\widetilde{\mathbf{x}} \ \left[\delta^{\mu\nu}\left(-\varphi^{*\prime}\partial_{\widetilde{x}^\mu} + \varphi^{*\prime} ie\widetilde{A}_\mu\right)(\widetilde{D}_\nu\varphi') + m^2\varphi^{*\prime}\varphi'\right]\right] \\
&= e^{S_E^{\mathbf{J}}[\boldsymbol{\varphi}_c]} \cdot \int D\varphi' \ D\varphi^{*\prime} \ \exp\left[-\frac{1}{2}\int d^4\widetilde{\mathbf{x}} \ \varphi^{*\prime}\left[-\delta^{\mu\nu}\widetilde{D}_\mu\widetilde{D}_\nu + m^2\right]\varphi'\right] \\
&= e^{S_E^{\mathbf{J}}[\boldsymbol{\varphi}_c]} \cdot \mathrm{Det}^{-1}\left[-\delta^{\mu\nu}\widetilde{D}_{\mu\nu}^2 + m^2\right] \cdot \mathrm{const},
\end{aligned} \tag{3.6}
$$

with the multiplicative constant depending solely on the integration measure. The Schwinger functional is thus given by

$$
W^{11}[J] := -i\ln Z[J] = -iS_E^J[\varphi_c] + i\ln\mathrm{Det}\left[-\delta^{\mu\nu}\widetilde{D}_{\mu\nu}^2 + m^2\right] + \mathrm{const}\,. \tag{3.7}
$$

We seek the Legendre transform Γ of W in the variable[8]

$$
\frac{dW}{d\mathbf{J}} = \underbrace{\left(\frac{\partial W}{\partial \mathbf{J}}\right)_{\boldsymbol{\varphi}_c}}_{\varphi_c} + \underbrace{\left(\frac{\partial W}{\partial \boldsymbol{\varphi}_c}\right)_{\mathbf{J}}}_{\left(\frac{\partial S_E^{\mathbf{J}}}{\partial \boldsymbol{\varphi}}\right)_{\mathbf{J}}(\boldsymbol{\varphi}_c)=0} \cdot \frac{\partial \boldsymbol{\varphi}_c}{\partial \mathbf{J}} = \boldsymbol{\varphi}_c \tag{3.8}
$$

and obtain the so called *effective action*[9]

$$
\Gamma[\boldsymbol{\varphi}_c] = W[\mathbf{J}] - \int d^4\mathbf{x}\ (J^*\varphi_c + \varphi_c^* J) \overset{(3.7)}{=} -iS_E^{\mathbf{J}}[\boldsymbol{\varphi}_c] + i\ln\mathrm{Det}\left[-\delta^{\mu\nu}\widetilde{D}_{\mu\nu}^2 + m^2\right] + \mathrm{const} \quad . \tag{3.9}
$$

Similarly, the effective action for a real scalar field φ with self-interaction potential $U(\varphi)$, is up to one-loop-order[10] given by[11]

$$
\Gamma^{11}[\varphi_c] = -iS_E^J[\varphi_c] + \frac{i}{2}\ln\mathrm{Det}\left[-\delta^{\mu\nu}\partial_{\mu\nu}^2 + m^2 + U''(\varphi_c)\right] + \mathrm{const} \quad . \tag{3.10}
$$

All our following efforts in this section, will concentrate on evaluating the 2nd. term (*1-loop correction*) in (3.9) and (3.10). We omit all "$\sim$" on fields and differential operators, while keeping in mind their actual nature[12].

3.2 The self-interacting case

3.2.1 Reduction to path integrals

Consider a Klein-Gordon field with self-interaction potential U. Then according to (3.10), the 1-loop-correction to the effective action[13] is given by

$$
\Gamma^{\mathrm{KG}}[\varphi] := \frac{i}{2}\ln\mathrm{Det}\left[-\delta^{\mu\nu}\partial_{\mu\nu}^2 + m^2 + U''(\varphi)\right] = \frac{i}{2}\ln\mathrm{Det}\Big[\underbrace{\hat{\mathbf{p}}^2 + m^2 + U''(\varphi)}_{=:\mathfrak{D}}\Big], \tag{3.11}
$$

---

[8] Note that this is no typo: The canonical variable $\frac{dW^{11}}{dJ}$ corresponding to J is indeed the classical field φ_c. This is due to the fact, that one-loop correction term $i\ln\mathrm{Det}\left[-\delta^{\mu\nu}\widetilde{D}_{\mu\nu}^2 + m^2\right]$ does not depend on $\boldsymbol{\varphi}$ or $\mathbf{J}$.

[9] Since W is the Legendre transform of Γ, one has $\frac{\delta\Gamma}{\delta\boldsymbol{\varphi}(\mathbf{x})} = -\mathbf{J}(\mathbf{x})$. On the other hand it can be shown that, $\frac{dW}{d\mathbf{J}(\mathbf{x})} = \langle 0|\,\hat{\boldsymbol{\varphi}}(\mathbf{x})\,|0\rangle_{\mathbf{J}}$. Thus, the field extremizing the effective action Γ, is exactly the vacuum expectation value $\langle 0|\,\hat{\boldsymbol{\varphi}}(\mathbf{x})\,|0\rangle_{\mathbf{J}=0}$. Hence, Γ can be thought of as describing the dynamics of vacuum expectation values.

[10] Note that the expansion (3.5) is no longer exact.

[11] Notice the additional factor $\times\frac{1}{2}$ in the real case, resulting from a different Gaussian integral than (3.6).

[12] See appendix A.5 on how to obtain the original (Minkowskian) effective action and N-point functions.

[13] In Euclidean form.

which together with $\ln \mathrm{Det}(\cdot) = \mathrm{Tr} \ln(\cdot)$ and identity

$$\mathrm{Tr} \ln A = -\int_0^\infty \frac{dT}{T} \mathrm{Tr}\, e^{-TA} \tag{3.12}$$

holding for positive operators A, implies

$$\Gamma^{\mathrm{KG}}[\varphi] = -\frac{i}{2} \int_0^\infty \frac{dT}{T} \int d^4\mathbf{x} \ \langle \mathbf{x}| \, e^{-T\mathfrak{D}} \, |\mathbf{x}\rangle \quad . \tag{3.13}$$

Interpreting

$$\mathfrak{D} = \hat{\mathbf{p}}^2 + m^2 + U''(\varphi(\hat{\mathbf{x}})) \tag{3.14}$$

as 4-dimensional Hamiltonian for a *particle* with *mass* $\mu = 1/2$, potential $V := m^2 + U''(\varphi)$ moving for the time $\widetilde{T} := -iT$, we may rewrite the transition element in (3.33) as described in (A.3):

$$\langle \mathbf{x}| \, e^{T\mathfrak{D}} \, |\mathbf{x}\rangle = \langle \mathbf{x}| \, e^{-it\mathfrak{D}} \, |\mathbf{x}\rangle = \int_{(t=0,\mathbf{x})}^{(t=\widetilde{T},\mathbf{x})} D\mathbf{x} \ \exp\left[i \int_0^{\widetilde{T}} dt \ \left[\frac{\mu}{2} (\partial_t \mathbf{x})^2 - V(\mathbf{x}) \right] \right]$$

$$\overset{\tau := it}{\equiv} \int_{(\tau=0,\mathbf{x})}^{(\tau=T,\mathbf{x})} D\mathbf{x} \ \exp\left[- \int_0^T d\tau \ \left[\frac{(\partial_\tau \mathbf{x})^2}{4} + m^2 + U''(\varphi(\mathbf{x})) \right] \right] \tag{3.15}$$

to obtain

$$\boxed{\Gamma^{\mathrm{KG}}[\varphi] = -\frac{i}{2} \int_0^\infty \frac{dT}{T} \oint D\mathbf{x} \ e^{-\int_0^T d\tau \ \mathcal{L}(\varphi;\mathbf{x},\dot{\mathbf{x}})},} \tag{3.16}$$

whereas

$$\boxed{\mathcal{L}(\varphi; \mathbf{x}, \dot{\mathbf{x}}) := \frac{\dot{\mathbf{x}}^2}{4} + m^2 + U''(\varphi).} \tag{3.17}$$

3.2.2 N-point functions for 3rd-order interaction

As is known[13], the one-particle irreducible N-point functions, can be obtained from the functional derivatives of the effective action Γ. Using the results of section 3.2.1, we can now attempt to to calculate the latest for the Klein-Gordon field. Consider the 1-loop-contribution

$$\Gamma[\varphi] = -\frac{i}{2} \int_0^\infty \frac{dT}{T} e^{-m^2 T} \oint D\mathbf{x} \ \exp\left[- \int_0^T \left[\frac{\dot{\mathbf{x}}^2}{4} + U''(\varphi) \right] \ d\tau \right] \tag{3.18}$$

to the effective action of a real Klein-Gordon field, as obtained in section 3.2.1. We shall calculate the functional derivatives

$$\Gamma_N[\mathbf{x}_1, .., \mathbf{x}_N] := \left. \frac{\delta^N \Gamma}{\delta\varphi(\mathbf{x}_1) \ldots \delta\varphi(\mathbf{x}_N)} \right|_{\varphi=0} \tag{3.19}$$

in the special case that $U(\varphi) = \frac{\lambda}{3!}\varphi^3$. Applying (3.19) to (3.18), we obtain

$$\Gamma_N[\mathbf{x}_1, .., \mathbf{x}_N] = -\frac{i}{2} (-\lambda)^N \int_0^\infty \frac{dT}{T} e^{-m^2 T} \int D\mathbf{x} \ e^{-\int_0^T \frac{\dot{\mathbf{x}}^2}{4} d\tau} \int_0^T \prod_{k=1}^N d\tau_k \ \delta(\mathbf{x}(\tau_k) - \mathbf{x}_k). \tag{3.20}$$

In momentum space, expression (3.20) becomes

$$\widetilde{\Gamma}_N[\mathbf{p}^1,..,\mathbf{p}^N] := (2\pi)^{-DN/2}\int d\mathbf{x}_1\dots d\mathbf{x}_N\ \Gamma_N[\mathbf{x}_1,..,\mathbf{x}_N]\cdot\exp\left[i\sum_{k=1}^N \mathbf{p}^k\mathbf{x}_k\right]$$

$$= -\frac{i}{2}\frac{(-\lambda)^N}{(2\pi)^{DN/2}}\int_0^\infty \frac{dT}{T}e^{-m^2T}\int_0^T d\tau_1..d\tau_N \oint D\mathbf{x}\ e^{-\int_0^T \frac{\dot{\mathbf{x}}^2}{4}d\tau}\cdot\exp\left[i\sum_{k=1}^N \mathbf{p}^k\cdot\mathbf{x}(\tau_k)\right]. \tag{3.21}$$

For trajectory $\mathbf{x}(\tau)$ we define $\overline{\mathbf{x}} := \frac{1}{T}\int_0^T \mathbf{x}(\tau)\ d\tau$ and $\mathbf{y}(\tau) := \mathbf{x}(\tau) - \overline{\mathbf{x}}$. With this, for any path-functional F the integral $\oint D\mathbf{x}\ F[\mathbf{x}]$ can be written as $\int d\overline{\mathbf{x}} \oint_{\overline{\mathbf{y}}=0} F[\overline{\mathbf{x}}+\mathbf{y}]$ and the path integral in (3.21) takes the form

$$\underbrace{\int d\overline{\mathbf{x}}\exp\left[i\sum_{k=1}^N \mathbf{p}^k\cdot\overline{\mathbf{x}}\right]}_{(2\pi)^D\delta\left[\sum_{k=1}^N \mathbf{p}^k\right]}\oint_{\overline{\mathbf{y}}=0} D\mathbf{y}\ e^{-\int_0^T \frac{\dot{\mathbf{y}}^2}{4}d\tau}\cdot\exp\left[i\sum_{k=1}^N \mathbf{p}^k\cdot\mathbf{y}(\tau_k)\right]. \tag{3.22}$$

Since boundary conditions $\mathbf{y}(0) = \mathbf{y}(T)$ and $\dot{\mathbf{y}}(0) = \dot{\mathbf{y}}(T)$ are assumed for the remaining path integral, we can write

$$\langle\dot{\mathbf{y}},\dot{\mathbf{y}}\rangle_T := \int_0^T d\tau\ \dot{\mathbf{y}} = -\int_0^T d\tau\ \mathbf{y}\ddot{\mathbf{y}} = \left\langle \mathbf{y}, -\frac{d^2}{d\tau^2}\mathbf{y}\right\rangle_T, \tag{3.23}$$

while by (A.2) the operator $\frac{d^2}{d\tau^2}$ is invertible on the class of paths $\mathbf{y}$ on $[0,T]$ with $\overline{\mathbf{y}} = 0$ and Green's function $G_B/2$, whereas

$$G_B(\tau_1,\tau_2) := |\tau_2-\tau_1| - \frac{(\tau_2-\tau_1)^2}{T}. \tag{3.24}$$

The remaining path integral in (3.22) can thus be evaluated as a Gaussian one to yield

$$\oint_{\overline{\mathbf{y}}=0} D\mathbf{y}\ \exp\left[-\left\langle \mathbf{y}, -\frac{1}{4}\frac{d^2}{d\tau^2}\mathbf{y}\right\rangle_T + \left\langle i\sum_{k=1}^N \mathbf{p}^k\cdot\delta(\tau_k-\cdot),\mathbf{y}\right\rangle_T\right]$$

$$=(4\pi T)^{-\frac{D}{2}}\exp\left[-\frac{4i^2}{4}\sum_{k,l=1}^N \mathbf{p}^k\mathbf{p}^l\left\langle\delta(\tau_k-\cdot),\left[\frac{d^2}{d\tau^2}\right]^{-1}\delta(\tau_l-\cdot)\right\rangle_T\right]$$

$$\stackrel{(3.24)}{=}(4\pi T)^{-\frac{D}{2}}\exp\left[\frac{1}{2}\sum_{k,l=1}^N \mathbf{p}^k\mathbf{p}^l G_B(\tau_k,\tau_l)\right], \tag{3.25}$$

whereas we used the convention[14]

$$\oint_{\overline{\mathbf{y}}=0} D\mathbf{y}\ e^{-\int_0^T d\tau\ \frac{\dot{\mathbf{y}}^2}{4}} = (4\pi T)^{-\frac{D}{2}}\quad . \tag{3.29}$$

[14] This convention is motivated by the fact that for a free D-dimensional particle with mass $m = 1/2$, the transition amplitude for periodic boundaries is given by

$$\langle 0,\mathbf{x}_0\,|t_\beta,\mathbf{x}_0\rangle = \int_{(0,\mathbf{x}_0)}^{(t_\beta,\mathbf{x}_0)} D\mathbf{z}\ \exp\left[i\int_0^{t_\beta}\frac{\dot{\mathbf{z}}^2}{4}\ dt\right] = (4\pi i t_\beta)^{-\frac{D}{2}}. \tag{3.26}$$

Replacing $T := it_\beta$ and $\tau := it$, this implies

$$\int_{(0,\mathbf{x}_0)}^{(T,\mathbf{x}_0)} D\mathbf{z}\ \exp\left[-\int_0^T\frac{\dot{\mathbf{z}}^2}{4}\ d\tau\right] = (4\pi T)^{-\frac{D}{2}}. \tag{3.27}$$

Thus, using (3.22) and (3.25), we can write (3.21) in the final form

$$\widetilde{\Gamma}_N[\mathbf{p}^1,..,\mathbf{p}^N] = -\frac{i}{2}\cdot\frac{\pi^{\frac{D}{2}}(-\lambda)^N}{(2\pi)^{DN/2}}\cdot\delta\left[\sum_{k=1}^{N}\mathbf{p}^k\right]\int_0^\infty \frac{dT}{T^{1+\frac{D}{2}}}e^{-m^2T}\int_0^T d\tau_1..d\tau_N\ \exp\left[\frac{1}{2}\sum_{k,l=1}^{N}\mathbf{p}^k\mathbf{p}^l G_B(\tau_k,\tau_l)\right]. \quad (3.30)$$

3.3 The charged case

3.3.1 Reduction to path integrals

Consider a complex Klein-Gordon field of charge e, in a background gauge field $\mathbf{A}$. Then according to (3.9), the 1-loop-correction to the effective action[15] is given by

$$\Gamma^{\mathrm{KG}} := i\ln\mathrm{Det}\Big[-\delta^{\mu\nu}D_{\mu\nu}^2+m^2\Big] = i\ln\mathrm{Det}\Big[\underbrace{(\hat{\mathbf{p}}-e\mathbf{A})^2+m^2}_{=:\mathfrak{D}}\Big], \quad (3.31)$$

which together with $\ln\mathrm{Det}(\cdot)=\mathrm{Tr}\ln(\cdot)$ and Frullani's integral identity[16]

$$\mathrm{Tr}\ln A = -\int_0^\infty \frac{dT}{T}\,\mathrm{Tr}\,e^{-TA} \quad (3.32)$$

implies

$$\Gamma^{\mathrm{KG}}[\varphi] = -i\int_0^\infty\frac{dT}{T}\int d^4\mathbf{x}\ \langle\mathbf{x}|\,e^{-T\mathfrak{D}}\,|\mathbf{x}\rangle\quad . \quad (3.33)$$

Interpreting

$$\mathfrak{D} = \hat{\mathbf{p}}^2 - \{e\hat{\mathbf{A}},\hat{\mathbf{p}}\} + e^2\hat{\mathbf{A}}^2 + m^2 \quad (3.34)$$

as 4-dimensional Hamiltonian for a *particle* with *mass* $\mu = 1/2$, potential $V := e^2\hat{\mathbf{A}}^2 + m^2$ moving for the time $\widetilde{T} := -iT$, we may rewrite the transition element in (3.33) as described in (A.3):

$$\langle\mathbf{x}|\,e^{T\mathfrak{D}}\,|\mathbf{x}\rangle = \langle\mathbf{x}|\,e^{-it\mathfrak{D}}\,|\mathbf{x}\rangle = \int_{(t=0,\mathbf{x})}^{(t=\widetilde{T},\mathbf{x})} D\mathbf{x}\ \exp\left[i\int_0^{\widetilde{T}} dt\ \left[\frac{\mu}{2}(\partial_t\mathbf{x})^2+2\mu e^2\mathbf{A}^2+2\mu(\partial_t\mathbf{x})e\mathbf{A}-V\right]\right]$$

$$\overset{\tau:=it}{=} \int_{(\tau=0,\mathbf{x})}^{(\tau=T,\mathbf{x})} D\mathbf{x}\ \exp\left[\int_0^T d\tau\ \left[-\frac{(\partial_\tau\mathbf{x})^2}{4}+i(\partial_\tau\mathbf{x})e\mathbf{A}-m^2\right]\right] \quad (3.35)$$

to obtain

$$\boxed{\Gamma^{\mathrm{KG}}[\varphi] = -i\int_0^\infty\frac{dT}{T}\oint D\mathbf{x}\ e^{-\int_0^T d\tau\ \mathcal{L}(\varphi;\mathbf{x},\dot{\mathbf{x}})},} \quad (3.36)$$

whereas

$$\boxed{\mathcal{L}(\varphi;\mathbf{x},\dot{\mathbf{x}}) := \frac{\dot{\mathbf{x}}^2}{4} - ie\dot{\mathbf{x}}\mathbf{A} + m^2.} \quad (3.37)$$

Note that this result is up to a multiplicative constant, formally the same as (4.34), when one *ignores* all Grassmann-valued contributions.

Also note that

$$\int d\mathbf{x}_0 \int_{(0,\mathbf{x}_0)}^{(T,\mathbf{x}_0)} D\mathbf{z}\ e^{-\int_0^T\frac{\dot{\mathbf{z}}^2}{4}d\tau} \cong \oint_0^T D\mathbf{x}\ e^{-\int_0^T\frac{\dot{\mathbf{x}}^2}{4}d\tau} \cong \int d\overline{\mathbf{x}}\oint_{\overline{\mathbf{y}}=\overline{\mathbf{x}}} D\mathbf{y}\ e^{-\int_0^T\frac{\dot{\mathbf{y}}^2}{4}d\tau}. \quad (3.28)$$

See Schubert[5] for more.

[15] In Euclidean form.

[16] Actually differing by a universal, additive constant which shall be omited in the following.

3.3.2 N-point functions

Consider the 1-loop-correction[17]

$$\Gamma[\mathbf{A}] = -i\int_0^\infty \frac{dT}{T}e^{-m^2T}\int D\mathbf{x}\;\exp\left[-\int_0^T d\tau\left(\frac{\dot{\mathbf{x}}^2}{4} - ie\mathbf{A}\dot{\mathbf{x}}\right)\right] \tag{3.38}$$

to the effective action of a non-self-interacting Klein-Gordon-field in a background gauge field $\mathbf{A}$, as obtained in section 3.3.1. We shall derive an expression for the functional derivatives of $\Gamma[\mathbf{A}]$ with respect to the field $\mathbf{A}$, evaluated at $\mathbf{A}=0$. Similar to section 3.2.2, we obtain[18] from (3.38)

$$\Gamma^{\mu_1..\mu_N}[\mathbf{x}_1,..,\mathbf{x}_N] := \left.\frac{\delta^N\Gamma}{\delta A_{\mu_1}(\mathbf{x}_1)..\delta A_{\mu_N}(\mathbf{x}_N)}\right|_{\mathbf{A}=0}$$

$$= -i(ie)^N\int_0^\infty \frac{dT}{T}e^{-m^2T}\oint D\mathbf{x}\; e^{-\int_0^T d\tau\frac{\dot{\mathbf{x}}^2}{4}}\prod_{k=1}^N\int_0^T d\tau_k\;\dot{x}^{\mu_k}(\tau_k)\delta(\mathbf{x}(\tau_k)-\mathbf{x}_k). \tag{3.39}$$

Taking the Fourier-transform[19] of (3.39) we obtain

$$\widetilde{\Gamma}_N\left[\mathbf{p}^1,..,\mathbf{p}^N;\boldsymbol{\varepsilon}^1,..,\boldsymbol{\varepsilon}^N\right] := \frac{\varepsilon^1_{\mu_1}}{(2\pi)^{\frac{D}{2}}}..\frac{\varepsilon^N_{\mu_N}}{(2\pi)^{\frac{D}{2}}}\int d\mathbf{x}_1..d\mathbf{x}_N\;\Gamma^{\mu_1..\mu_N}[\mathbf{x}_1,..,\mathbf{x}_N]\;e^{i\mathbf{p}^1\mathbf{x}_1}\dots e^{i\mathbf{p}^N\mathbf{x}_N}$$

$$= -\frac{i(ie)^N}{(2\pi)^{DN/2}}\int_0^\infty\frac{dT}{T}e^{-m^2T}\oint D\mathbf{x}\;e^{-\int_0^T d\tau\frac{\dot{\mathbf{x}}^2}{4}}\prod_{k=1}^N\int_0^T d\tau_k\;\boldsymbol{\varepsilon}^k\dot{\mathbf{x}}(\tau_k)e^{i\mathbf{p}^k\mathbf{x}(\tau_k)}. \tag{3.40}$$

In analogy to section 3.2.2, we separate the path-integral $\oint D\mathbf{x}$ into $\int d\overline{\mathbf{x}}\oint_{\overline{\mathbf{y}}=0} D\mathbf{y}$, while keeping in mind that (3.23) holds. Furthermore, we shall write $\boldsymbol{\varepsilon}^k\dot{\mathbf{x}} =: e^{\boldsymbol{\varepsilon}^k\mathbf{x}}\big|_{\substack{\text{lin in}\\ \text{each }\boldsymbol{\varepsilon}^k}}$ while keeping in mind, that at the end only terms linear in each $\boldsymbol{\varepsilon}^k$ are to be taken. We can thus write the path integral of (3.40) as

$$\oint D\mathbf{x}\;e^{-\int_0^T\frac{\dot{\mathbf{x}}^2}{4}d\tau}\prod_{k=1}^N\boldsymbol{\varepsilon}^k\dot{\mathbf{x}}(\tau_k)e^{i\mathbf{p}^k\mathbf{x}(\tau_k)} = \underbrace{\int d\overline{\mathbf{x}}\;e^{i\overline{\mathbf{x}}(\mathbf{p}^1+..+\mathbf{p}^N)}}_{(2\pi)^D\delta\left[\sum_{k=1}^N\mathbf{p}^k\right]}\cdot\oint_{\overline{\mathbf{y}}=0}e^{-\int_0^T\frac{\dot{\mathbf{y}}^2}{4}d\tau}\prod_{k=1}^N\boldsymbol{\varepsilon}^k\dot{\mathbf{y}}(\tau_k)e^{i\mathbf{p}^k\mathbf{y}(\tau_k)}$$

$$= (2\pi)^D\delta\left[\sum_{k=1}^N\mathbf{p}^k\right]\oint_{\overline{\mathbf{y}}=0}D\mathbf{y}\;\exp\left[-\int_0^T\frac{\dot{\mathbf{y}}^2}{4}d\tau + \sum_{k=1}^N\boldsymbol{\varepsilon}^k\dot{\mathbf{y}}(\tau_k) + i\mathbf{p}^k\mathbf{y}(\tau_k)\right]\Bigg|_{\substack{\text{lin in}\\ \text{each }\boldsymbol{\varepsilon}^k}}$$

$$\overset{(3.23)}{=} (2\pi)^D\delta\left[\sum_{k=1}^N\mathbf{p}^k\right]\oint_{\overline{\mathbf{y}}=0}D\mathbf{y}\;\exp\left[-\left\langle\mathbf{y},-\frac{1}{4}\frac{d^2}{d\tau^2}\mathbf{y}\right\rangle_T + \langle\mathbf{J},\mathbf{y}\rangle_T\right]\Bigg|_{\substack{\text{lin in}\\ \text{each }\boldsymbol{\varepsilon}^k}}, \tag{3.41}$$

whereas

$$\mathbf{J} := \sum_{k=1}^N\delta(\tau_k-\cdot)\left[\boldsymbol{\varepsilon}^k\partial_\tau + i\mathbf{p}^k\right] \tag{3.42}$$

[17] In Euclidean form.
[18] Note that $\Gamma^{\mu_1..\mu_N}$ is of tensorial nature.
[19] With respect to the orthonormal system $\left|\mathbf{p}_1,..,\mathbf{p}_N;\boldsymbol{\varepsilon}^1,..,\boldsymbol{\varepsilon}^N\right\rangle$, with $\boldsymbol{\varepsilon}^k$ as *polarization* vector.

can be thought of as a *source*, acting as linear functional on the paths by means of $\langle J, \cdot\rangle_T$. Similar to section 3.2.2, we can formally evaluate (3.41) as a Gaussian integral, yielding

$$(2\pi)^D \cdot \delta\left[\sum_{k=1}^{N} \mathbf{p}^k\right] \cdot (4\pi T)^{-\frac{D}{2}} \exp\left[\left\langle \mathbf{J}, \left(-\tfrac{d^2}{d\tau^2}\right)^{-1} \mathbf{J}\right\rangle_T\right]\Bigg|_{\substack{\text{lin in}\\ \text{each } \boldsymbol{\varepsilon}^k}}$$

$$= \left[\frac{\pi}{T}\right]^{\frac{D}{2}} \cdot \delta\left[\sum_{k=1}^{N} \mathbf{p}^k\right] \exp\left[-\sum_{k,l=1}^{N} (\boldsymbol{\varepsilon}^k\partial_1 + i\mathbf{p}^k)(\boldsymbol{\varepsilon}^l\partial_2 + i\mathbf{p}^l) \underbrace{\left\langle \delta(\tau_k - \cdot), \left(\tfrac{d^2}{d\tau^2}\right)^{-1} \delta(\tau_l - \cdot)\right\rangle_T}_{\frac{1}{2}G_B(\tau_k,\tau_l)}\right]\Bigg|_{\substack{\text{lin in}\\ \text{each } \boldsymbol{\varepsilon}^k}}$$

$$= \left[\frac{\pi}{T}\right]^{\frac{D}{2}} \cdot \delta\left[\sum_{k=1}^{N} \mathbf{p}^k\right] \exp\left[\frac{1}{2}\sum_{k,l=1}^{N} \left[\mathbf{p}^k\mathbf{p}^l - i\mathbf{p}^l\boldsymbol{\varepsilon}^k\partial_1 - i\mathbf{p}^k\boldsymbol{\varepsilon}^l\partial_2 - \boldsymbol{\varepsilon}^k\boldsymbol{\varepsilon}^l\partial_{1,2}^2\right] G_B(\tau_k,\tau_l)\right]\Bigg|_{\substack{\text{lin in}\\ \text{each } \boldsymbol{\varepsilon}^k}}, \tag{3.43}$$

whereas use of the convention (3.29) has been made. Here $\partial_1 G_B$ and $\partial_2 G_B$ denote derivatives with respect to the 1st and 2nd argument of G_B. Note that[20]

$$\begin{aligned} &G_B(x,y) = G_B(y,x),\\ &\partial_1 G_B = -\partial_2 G_B,\\ &\partial_i G_B(x,y) = -\partial_i G_B(y,x) \quad , \quad i \in \{1,2\},\\ &\partial_{ij} G_B(x,y) = \partial_{ij} G_B(y,x) \quad , \quad i,j \in \{1,2\}. \end{aligned} \tag{3.44}$$

Inserting (3.43) back into (3.40), finally yields

$$\widetilde{\Gamma}_N[\mathbf{p}^1, \boldsymbol{\varepsilon}^1, .., \mathbf{p}^N, \boldsymbol{\varepsilon}^N] = -\,i\frac{\pi^{\frac{D}{2}}(ie)^N}{(2\pi)^{DN/2}} \cdot \delta\left[\sum_{k=1}^{N} \mathbf{p}^k\right] \cdot \int_0^\infty \frac{dT}{T^{1+\frac{D}{2}}} e^{-m^2 T} \prod_{k=1}^{N}\left[\int_0^T d\tau_k\right]$$

$$\times \exp\left[\frac{1}{2}\sum_{k,l=1}^{N}\left[\mathbf{p}^k\mathbf{p}^l G_B(\tau_k,\tau_l) - 2i\boldsymbol{\varepsilon}^k\mathbf{p}^l\partial_1 G_B(\tau_k,\tau_l) + \boldsymbol{\varepsilon}^k\boldsymbol{\varepsilon}^l\partial_1^2 G_B(\tau_k,\tau_l)\right]\right]\Bigg|_{\substack{\text{lin in}\\ \text{each } \boldsymbol{\varepsilon}^k}}. \tag{3.45}$$

3.3.3 The vacuum polarization tensor

Specializing (3.45) to the case $N = 2$, one obtains

$$\widetilde{\Gamma}_2[\mathbf{p}^1, \mathbf{p}^2; \boldsymbol{\varepsilon}^1, \boldsymbol{\varepsilon}^2] = -\,i\frac{\pi^{\frac{D}{2}}(ie)^2}{(2\pi)^D} \cdot \delta\left[\sum_{k=1}^{N} \mathbf{p}^k\right] \cdot \int_0^\infty \frac{dT}{T^{1+\frac{D}{2}}} e^{-m^2 T} \int_0^T\int_0^T d\tau_1\, d\tau_2$$

$$e^{\mathbf{p}^1\mathbf{p}^2 G_B(\tau_1,\tau_2)} \cdot \left[(\boldsymbol{\varepsilon}^1\mathbf{p}^2)(\boldsymbol{\varepsilon}^2\mathbf{p}^1)\left(\partial_1 G_B(\tau_1,\tau_2)\right)^2 + \boldsymbol{\varepsilon}^1\boldsymbol{\varepsilon}^2\partial_1^2 G_B(\tau_1,\tau_2)\right]. \tag{3.46}$$

[20]From A.2 we know that $G_B(\tau_1,\tau_2) = |\tau_1 - \tau_2| - \frac{(\tau_1-\tau_2)^2}{T}$. In particular

$$\partial_\tau G_B(\tau_1,\tau_2) = \text{sgn}(\tau_1 - \tau_2) - \frac{2}{T}(\tau_1 - \tau_2)$$

and

$$\partial_\tau^2 G_B(\tau_1,\tau_2) = 2\delta(\tau_1 - \tau_1) - \frac{2}{T} \quad .$$

The polarization tensor[21] is thus given by

$$\widetilde{\Pi}^{\mu\nu}[\mathbf{p}] = -\frac{i}{(2\pi)^{\frac{D}{2}}}\frac{\pi^{\frac{D}{2}}(ie)^2}{(2\pi)^D}\int_0^\infty \frac{dT}{T^3}e^{-m^2T}\int_0^T\int_0^T d\tau_1 d\tau_2\ e^{-(\mathbf{p})^2 G_B(\tau_1,\tau_2)}$$

$$\times\Big[\delta_{\mu\nu}\partial_\tau^2 \underbrace{G_B(\tau_1,\tau_2)}_{G_B(\tau_1-\tau_2)} - p_\mu p_\nu \left(\partial_\tau G_B(\tau_1,\tau_2)\right)^2\Big]. \tag{3.47}$$

Using the fact that G_B and its derivatives are defined on the circle $\mathbb{R}/[0,T]$, we replace $\int_0^T\int_0^T f(G_B(\tau_1-\tau_2))d\tau_2 d\tau_1$ with $\int_0^T\int_{-\tau_2}^{T-\tau_2} G_B(\tau)d\tau d\tau_2 = \int_0^T\int_0^T G_B(\tau)d\tau d\tau_2$ and obtain

$$\widetilde{\Pi}^{\mu\nu}[\mathbf{p}] = -\frac{i}{(2\pi)^{\frac{D}{2}}}\frac{\pi^{\frac{D}{2}}(ie)^2}{(2\pi)^D}\int_0^\infty \frac{dT}{T^{\frac{D}{2}}}e^{-m^2T}\int_0^T d\tau\ e^{-(\mathbf{p})^2 G_B(\tau)}\Big[\delta_{\mu\nu}\partial_\tau^2 G_B(\tau) - p_\mu p_\nu\left(\partial_\tau G_B(\tau)\right)^2\Big]. \tag{3.48}$$

Performing a partial integration on the integrant $e^{(\mathbf{p})^2 G_B}\partial_\tau^2 G_B$ in (3.48) and substituting $u := \tau/T$, results in

$$\widetilde{\Pi}^{\mu\nu}[\mathbf{p}] = -\frac{i}{(2\pi)^{\frac{D}{2}}}\frac{(ie)^2}{(4\pi)^{\frac{D}{2}}}\int_0^\infty \frac{dT}{T^{\frac{D}{2}}}e^{-m^2T}\int_0^T d\tau\ e^{-(\mathbf{p})^2 G_B(\tau)}\Big[(\mathbf{p})^2\delta_{\mu\nu} - p_\mu p_\nu\Big]\left(\partial_\tau G_B(\tau)\right)^2$$

$$= -\frac{i}{(2\pi)^{\frac{D}{2}}}\frac{e^2}{(4\pi)^{\frac{D}{2}}}\Big[p_\mu p_\nu - (\mathbf{p})^2\delta_{\mu\nu}\Big]\int_0^\infty \frac{dT}{T^{\frac{D}{2}-1}}e^{-m^2T}\int_0^1 du\ e^{-(\mathbf{p})^2 uT(1-u)}\big(\underbrace{\mathrm{sgn}(u)}_{1} - 2u\big)^2$$

$$= -\frac{i}{(2\pi)^{\frac{D}{2}}}\frac{e^2}{(4\pi)^{\frac{D}{2}}}\Big[p_\mu p_\nu - (\mathbf{p})^2\delta_{\mu\nu}\Big]\Gamma\left(2-\tfrac{D}{2}\right)\int_0^1 du\ (1-2u)^2\left[m^2 + (\mathbf{p})^2 u(1-u)\right]^{\frac{D}{2}-2}\quad . \tag{3.49}$$

4 Worldline formalism for Dirac-fields

We shall, similar to section 3 for Klein-Gordon fields, derive a worldline formulation for the effective action of Dirac-fields and the corresponding N-point functions. The techniques, mainly taken from Schubert[5] and Mamedov[14], are similar in principle, with the difference lying in the anticommuting character of the fields used. This makes the usage of Grassmann algebras and integrals a necessity. For more information on the latter, see Berezin[8], Hoker[15] and Swanson[1].

4.1 The effective action

Consider the generating functional

$$Z[\eta,\overline{\eta}] := \int D\psi\ D\overline{\psi}\ \exp\Big[i\underbrace{\int d^4\mathbf{x}\ \mathcal{L}^{\mathrm{Dir}}(\psi,\overline{\psi},\partial\psi,\partial\overline{\psi})}_{=:S[\psi,\overline{\psi}]} + i\int(\overline{\eta}\psi + \overline{\psi}\eta)\ d^4\mathbf{x}\Big] \tag{4.1}$$

for the Dirac-field with Lagrangian

$$\mathcal{L}^{\mathrm{Dir}} = i\overline{\psi}\gamma^\mu D_\mu\psi - m\overline{\psi}\psi, \tag{4.2}$$

[21] Given by the relation $\widetilde{\Gamma}_2^{\mu\nu}[\mathbf{p}^1,\mathbf{p}^2] =: (2\pi)^{\frac{D}{2}}\cdot\widetilde{\Pi}_{\mu\nu}[\mathbf{p}^1]\cdot\delta[\mathbf{p}^1+\mathbf{p}^2]$. Note that this may differ by a factor of $(2\pi)^{\frac{D}{2}}$ to standard literature, as here the Fourier-transform is chosen to be the isometric one. See also appendix A.7.

whereas $D_\mu := (\partial_\mu - ieA_\mu)$. We perform a Wick-rotation to imaginary times $\widetilde{x}^0 := ix^0,\ \widetilde{x}^j := x^j$ and consider all fields as functions of $\widetilde{\mathbf{x}}$, so that Z becomes

$$Z[\eta,\overline{\eta}] = \int D\psi\ D\overline{\psi}\ \exp\Big[\underbrace{-iS_E[\psi,\overline{\psi}] - \int d^4\widetilde{\mathbf{x}}\ (\overline{\eta}\psi + \overline{\psi}\eta)}_{=:-iS_E^{\eta,\overline{\eta}}[\psi,\overline{\psi}]}\Big], \tag{4.3}$$

whereas

$$S_E[\psi,\overline{\psi}] = \int d^4\widetilde{\mathbf{x}}\ \mathcal{L}_E^{\mathrm{Dir}}(\psi,\overline{\psi},\widetilde{\partial}\psi,\widetilde{\partial}\overline{\psi}), \tag{4.4}$$

$$\mathcal{L}_E^{\mathrm{Dir}} := -i\overline{\psi}\widetilde{\gamma}^\mu\widetilde{D}_\mu\psi - im\overline{\psi}\psi,$$

$$\widetilde{D}_\mu := (\partial_{\widetilde{x}^\mu} - ie\widetilde{A}_\mu)\quad,\quad \widetilde{\mathbf{A}} := (-iA_0,\vec{A})\quad,\quad \widetilde{\gamma}^0 := \gamma^0,\ \widetilde{\gamma}^j := -i\gamma^j.$$

Let $\psi_c,\overline{\psi}_c$ extremize the action $S_E^{\eta,\overline{\eta}}$ and for every field configuration $\psi,\overline{\psi}$ set $\psi' := \psi-\psi_c,\ \overline{\psi}' := \overline{\psi}-\overline{\psi}_c$. Then up to second order in $\psi',\overline{\psi}'$ (*one-loop correction*)[22] we can write[23]

$$S_E^{\eta,\overline{\eta}}[\psi,\overline{\psi}] = S_E^{\eta,\overline{\eta}}[\psi_c,\overline{\psi}_c] + \underbrace{\frac{dS_E^{\eta,\overline{\eta}}}{d(\psi,\overline{\psi})}\Big|_{\mathrm{c}}}_{0}(\psi',\overline{\psi}') + (\psi',\overline{\psi}')\frac{1}{2!}\frac{d^2S_E^{\eta,\overline{\eta}}}{d(\psi,\overline{\psi})^2}\Big|_{\mathrm{c}}(\psi',\overline{\psi}'), \tag{4.5}$$

where $\big|_{\mathrm{c}}$ means evaluation at $\psi_c,\overline{\psi}_c$. Thus, (4.3) becomes

$$\begin{aligned}Z[\eta,\overline{\eta}] &= \exp\Big[-iS_E^{\eta,\overline{\eta}}[\psi_c,\overline{\psi}_c]\Big]\cdot\int D\psi'\ D\overline{\psi}'\ \exp\Big[-i(\psi',\overline{\psi}')\frac{1}{2!}\frac{d^2S_E^{\eta,\overline{\eta}}}{d(\psi,\overline{\psi})^2}\Big|_{\mathrm{c}}(\psi',\overline{\psi}')\Big]\\ &= \exp\Big[-iS_E^{\eta,\overline{\eta}}[\psi_c,\overline{\psi}_c]\Big]\cdot\int D\psi'\ D\overline{\psi}'\ \exp\Big[-i\int d^4\widetilde{\mathbf{x}}\ \overline{\psi}'\Big(-i\widetilde{\gamma}^\mu\widetilde{D}_\mu - im\Big)\psi'\Big]\\ &= \exp\Big[-iS_E^{\eta,\overline{\eta}}[\psi_c,\overline{\psi}_c]\Big]\cdot\det\Big[-i\widetilde{\gamma}^\mu\widetilde{D}_\mu - im\Big]\cdot\mathrm{const}\quad.\end{aligned} \tag{4.6}$$

The multiplicative constant at the end depends on the exact functional measure used. The Schwinger-function becomes

$$W[\eta,\overline{\eta}] := -i\ln Z[\eta,\overline{\eta}] = -S_E^{\eta,\overline{\eta}}[\psi_c,\overline{\psi_c}] - i\ln\det\Big[-i\widetilde{\gamma}^\mu\widetilde{D}_\mu - im\Big] + \mathrm{const}\quad. \tag{4.7}$$

Note that $\boldsymbol{\psi}_c := (\overline{\psi}_c,\psi_c)$ actually still depends on $\boldsymbol{\eta} := (\eta,\overline{\eta})$. We seek the Legendre transform Γ of W in the variable[24]

$$\frac{dW}{d\boldsymbol{\eta}} = \left(\frac{\partial W}{\partial\boldsymbol{\eta}}\right)_{\boldsymbol{\psi}_c} + \underbrace{\left(\frac{\partial W}{\partial\boldsymbol{\psi}_c}\right)_{\boldsymbol{\eta}}}_{\left(\frac{\partial S_E^{\boldsymbol{\eta}}}{\partial\boldsymbol{\psi}}\right)_{\boldsymbol{\eta}}(\boldsymbol{\psi}_c)=0}\cdot\frac{\partial\boldsymbol{\psi}_c}{\partial\boldsymbol{\eta}} = (\overline{\psi}_c,\psi_c) = \boldsymbol{\psi}_c \tag{4.8}$$

and obtain

$$\begin{aligned}\Gamma[\boldsymbol{\psi}_c] = W - \langle\boldsymbol{\eta},\boldsymbol{\psi}_c\rangle &\overset{(4.7)}{=} \underbrace{-S_E^{\boldsymbol{\eta}}[\boldsymbol{\psi}_c] - \int d^4\mathbf{x}\ (\overline{\eta}\psi_c + \overline{\psi}_c\eta)}_{-S_E[\boldsymbol{\psi}_c]=S[\boldsymbol{\psi}_c]} - i\ln\det\Big[-i\widetilde{\gamma}^\mu\widetilde{D}_\mu - im\Big] + \mathrm{const}\\ &= S[\boldsymbol{\psi}_c] - i\ln\det\Big[-i\widetilde{\gamma}^\mu\widetilde{D}_\mu - im\Big] + \mathrm{const}\,.\end{aligned} \tag{4.9}$$

[22] Which in the case of the Lagrangian (4.2), is indeed exact.

[23] With $\frac{d^2S_E^{\eta,\overline{\eta}}[\psi,\overline{\psi}]}{d(\psi,\overline{\psi})^2}$ as bilinear functional on fields $(\psi',\overline{\psi}')$.

[24] Note that this is no typo: The canonical variable $\frac{dW}{d\boldsymbol{\eta}}$[11] corresponding to $\boldsymbol{\eta}$ is indeed the classical field $\boldsymbol{\psi}_c$. This is due to the fact that the one-loop correction term $-i\ln\det\Big[-i\widetilde{\gamma}^\mu\widetilde{D}_\mu - im\Big]$ does not depend on $\boldsymbol{\psi}$.

We shall now concentrate on calculating the 2nd term (*1-loop correction*) of the effective action in (4.9), while omitting the "∼" on all fields and differential operators[25]:

$$\Gamma[\mathbf{A}] := -i \ln\det\left[\widetilde{\gamma}^{\mu}(\hat{p}_{\mu} - eA_{\mu}) - im\right]. \tag{4.10}$$

We note that[5]

$$\ln\mathrm{Det}\left[(\widetilde{\gamma}^{\mu}\hat{p}_{\mu} - e\widetilde{\gamma}^{\mu}A_{\mu}) - im\right] = \frac{1}{2}\ln\mathrm{Det}\left[(\widetilde{\gamma}^{\mu}\hat{p}_{\mu} - e\widetilde{\gamma}^{\mu}A_{\mu})^2 + m^2\right]. \tag{4.11}$$

Using $\ln\mathrm{Det}\,[\cdot] = \mathrm{Tr}\ln[\cdot]$ and formula (3.32), we conclude

$$\Gamma[\mathbf{A}] = -\frac{i}{2}\,\mathrm{Tr}\ln\left[(\widetilde{\gamma}^{\mu}\hat{p}_{\mu} - e\widetilde{\gamma}^{\mu}A_{\mu})^2 + m^2\right] = \frac{i}{2}\int_0^{\infty}\frac{dT}{T}\,\mathrm{Tr}\, e^{-T\mathfrak{D}[\hat{\mathbf{p}},\mathbf{A},(\widetilde{\gamma}_{\mu}\widetilde{\gamma}_{\nu})]} \quad , \tag{4.12}$$

whereas

$$\begin{aligned}\mathfrak{D}[\hat{\mathbf{p}},\mathbf{A},(\widetilde{\gamma}_{\mu}\widetilde{\gamma}_{\nu})] &:= \left(\widetilde{\gamma}^{\mu}\hat{p}_{\mu} - e\widetilde{\gamma}^{\mu}A_{\mu}\right)^2 + m^2 \\ &\overset{(A.1)}{=} (\hat{\mathbf{p}} - e\mathbf{A})^2\cdot\mathbb{1} + \frac{ie}{4}\left[\widetilde{\gamma}^{\mu},\widetilde{\gamma}^{\nu}\right]F_{\mu\nu} + m^2\cdot\mathbb{1} \quad .\end{aligned} \tag{4.13}$$

4.2 Reducing the effective action to path-integrals

We shall now further evaluate expression (4.12), seeking a worldline formulation similar to the scalar case in section 3. We proceed in 4 dimensions, but will later on generalize the result to $D = 4-\varepsilon$, in view of dimensional regularization. Consider the matrices

$$a_1^{\pm} := \frac{1}{2}(\widetilde{\gamma}^1 \pm i\widetilde{\gamma}^3) \quad , \quad a_2^{\pm} := \frac{1}{2}(\widetilde{\gamma}^2 \pm i\widetilde{\gamma}^0) \quad . \tag{4.14}$$

As they satisfy the anticommutation rules

$$\left\{a_r^+, a_s^-\right\} = \delta_{rs} \quad , \quad \left\{a_r^+, a_s^+\right\} = \left\{a_r^-, a_s^-\right\} = 0 \quad , \tag{4.15}$$

they can be interpreted as creation and annihilation operators on a Hilbert-space with a vacuum $|0\rangle \in \mathrm{kernel}\,\{a_r^-\}$. We introduce the real Grassmann variables[26] $\vartheta^1, \vartheta^2, \overline{\vartheta}^1, \overline{\vartheta}^2$ anticommuting with the $a_{1,2}^{\pm}$ and commuting with the vacuum $|0\rangle$. The states

$$\begin{aligned} |\boldsymbol{\vartheta}\rangle &:= \exp\left[-\vartheta^1 a_1^+ - \vartheta^2 a_2^+\right]|0\rangle & \left|\overline{\boldsymbol{\vartheta}}\right\rangle &:= i(\overline{\vartheta}^1 - a_1^+)(\overline{\vartheta}^2 - a_2^+)\,|0\rangle \\ \langle\boldsymbol{\vartheta}| &:= i\,\langle 0|\,(\vartheta^1 - a_1^-)(\vartheta^2 - a_2^-) & \left\langle\overline{\boldsymbol{\vartheta}}\right| &:= \langle 0|\exp\left[-a_1^-\overline{\vartheta}^1 - a_2^-\overline{\vartheta}^2\right] \end{aligned} \tag{4.16}$$

satisfy

$$\begin{aligned} \langle\boldsymbol{\vartheta}|\,a_r^- &= \langle\boldsymbol{\vartheta}|\,\vartheta^r & a_r^-\,|\boldsymbol{\vartheta}\rangle &= \vartheta^r\,|\boldsymbol{\vartheta}\rangle \\ \left\langle\overline{\boldsymbol{\vartheta}}\right|a_r^+ &= \left\langle\overline{\boldsymbol{\vartheta}}\right|\overline{\vartheta}^r & a_r^+\left|\overline{\boldsymbol{\vartheta}}\right\rangle &= \vartheta^r\left|\overline{\boldsymbol{\vartheta}}\right\rangle \\ \left\langle\boldsymbol{\vartheta}\,\middle|\overline{\boldsymbol{\vartheta}}\right\rangle &= \exp\big[\underbrace{\vartheta^1\overline{\vartheta}^1 + \vartheta^2\overline{\vartheta}^2}_{\boldsymbol{\vartheta}\cdot\overline{\boldsymbol{\vartheta}}}\big] & \left\langle\overline{\boldsymbol{\vartheta}}\,\middle|\boldsymbol{\vartheta}\right\rangle &= \exp\left[\overline{\boldsymbol{\vartheta}}\boldsymbol{\vartheta}\right] \end{aligned} \tag{4.17}$$

and are the coherent states of the Fock-space[27] and $\mathcal{H}_a$ induced by the $a_{1,2}^{\pm}$. In particular[28]

$$\mathbb{1}_{\mathcal{H}_a} = i\int d^2\boldsymbol{\vartheta}\;\;|\boldsymbol{\vartheta}\rangle\langle\boldsymbol{\vartheta}| = -i\int d^2\overline{\boldsymbol{\vartheta}}\;\;\left|\overline{\boldsymbol{\vartheta}}\right\rangle\left\langle\overline{\boldsymbol{\vartheta}}\right| \tag{4.18}$$

---

[25] None the less keeping in mind their true nature.

[26] See Berezin[8]. We shall write $\boldsymbol{\vartheta} := (\vartheta^1, \vartheta^2)$, $\overline{\boldsymbol{\vartheta}} := (\overline{\vartheta}^1, \overline{\vartheta}^2)$ and $\boldsymbol{\vartheta}\cdot\overline{\boldsymbol{\vartheta}} := \vartheta^1\overline{\vartheta}^1 + \vartheta^2\overline{\vartheta}^2$.

[27] See Glauber[17] and Klauder[18] for more on coherent states.

[28] The convention followed here for Grassmann-integrals shall be $\int\, d\vartheta^i\ \vartheta^i = \int d\overline{\vartheta}^i\ \overline{\vartheta}^i = -i$ and

$$\int \underbrace{d\vartheta^1 d\vartheta^2}_{d^2\boldsymbol{\vartheta}}\ \vartheta^1\vartheta^2 = (-i)^2 = \int \underbrace{d\overline{\vartheta}^2 d\overline{\vartheta}^1}_{d^2\overline{\boldsymbol{\vartheta}}}\ \overline{\vartheta}^2\overline{\vartheta}^1 \quad .$$

The anticommuting differentials $d\vartheta^i$, $d\overline{\vartheta}^i$ commute with the vacuum, but anticommute with all Grassmann variables and the $a_r^{\pm}$. For more information see Berezin[8].

and hence

$$\mathbb{1} = i \int d^4\mathbf{x}\; d^2\boldsymbol{\vartheta}\;\; |\mathbf{x},\boldsymbol{\vartheta}\rangle\, \langle\mathbf{x},\boldsymbol{\vartheta}| \quad . \tag{4.19}$$

The trace of an operator U in $\mathcal{H}_a$ is given by

$$\mathrm{Tr}\, U = i \int d^2\boldsymbol{\vartheta}\;\; \langle -\boldsymbol{\vartheta}|\, U\, |\boldsymbol{\vartheta}\rangle \quad . \tag{4.20}$$

For a derivation of equations (4.18) and (4.20), see appendix A.4. Consequently, for any $N \in \mathbb{N}$ we can write

$$\begin{aligned}
\mathrm{Tr}\, e^{-T\mathfrak{D}} &\overset{(4.20)}{=} i \int d^4\mathbf{x}\; d^2\boldsymbol{\vartheta}\;\; \langle \mathbf{x}, -\boldsymbol{\vartheta}|\, e^{-T\mathfrak{D}}\, |\mathbf{x},\boldsymbol{\vartheta}\rangle \\
&\overset{(4.19)}{=} i \int d^4\mathbf{x}\; d^2\boldsymbol{\vartheta}\; i^{N-1} \int \prod_{k=2}^{N} d^4\mathbf{x}_k\;\, d^2\boldsymbol{\vartheta}_k\;\; \langle \mathbf{x}, -\boldsymbol{\vartheta}|\, e^{-\frac{T}{N}\mathfrak{D}}\, |\mathbf{x}_N,\boldsymbol{\vartheta}_N\rangle \ldots \langle \mathbf{x}_2,\boldsymbol{\vartheta}_2|\, e^{-\frac{T}{N}\mathfrak{D}}\, |\mathbf{x},\boldsymbol{\vartheta}\rangle \\
&= i^N \int \prod_{k=1}^{N} d^4\mathbf{x}_k\;\, d^2\boldsymbol{\vartheta}_k\;\; \langle \mathbf{x}_{k+1},\boldsymbol{\vartheta}_{k+1}|\, e^{-\frac{T}{N}\mathfrak{D}}\, |\mathbf{x}_k,\boldsymbol{\vartheta}_k\rangle \quad \Bigg| \quad \mathbf{x}_{N+1} := \mathbf{x}_1 \;,\; \boldsymbol{\vartheta}_{N+1} := -\boldsymbol{\vartheta}_1 .
\end{aligned} \tag{4.21}$$

Using (4.18) we note that

$$\begin{aligned}
\langle \boldsymbol{\vartheta}_{k+1}|\, \widetilde{\gamma}_\mu \widetilde{\gamma}_\nu\, |\boldsymbol{\vartheta}_k\rangle &= -i \int d^2\overline{\boldsymbol{\vartheta}}_k\, \langle \boldsymbol{\vartheta}_{k+1}|\, \widetilde{\gamma}_\mu\, \big|\overline{\boldsymbol{\vartheta}}_k\big\rangle\, \big\langle \overline{\boldsymbol{\vartheta}}_k\big|\, \widetilde{\gamma}_\nu\, |\boldsymbol{\vartheta}_k\rangle \\
&= (..) = -i \int d^2\overline{\boldsymbol{\vartheta}}_k\, \langle \boldsymbol{\vartheta}_{k+1}\, \big|\overline{\boldsymbol{\vartheta}}_k\big\rangle\, \big\langle \overline{\boldsymbol{\vartheta}}_k\; |\boldsymbol{\vartheta}_k\rangle \cdot 2 \cdot {}_k\psi^\mu \cdot \psi_k^\nu
\end{aligned} \tag{4.22}$$

whereas

$${}_k\psi^\mu := \frac{\langle \boldsymbol{\vartheta}_{k+1}|\, \widetilde{\gamma}_\mu\, \big|\overline{\boldsymbol{\vartheta}}_k\big\rangle}{\sqrt{2}\, \langle \boldsymbol{\vartheta}_{k+1}\, \big|\overline{\boldsymbol{\vartheta}}_k\big\rangle} \quad , \quad \psi_k^\mu := \frac{\big\langle \overline{\boldsymbol{\vartheta}}_k\big|\, \widetilde{\gamma}_\mu\, |\boldsymbol{\vartheta}_k\rangle}{\sqrt{2}\, \big\langle \overline{\boldsymbol{\vartheta}}_k\; |\boldsymbol{\vartheta}_k\rangle} . \tag{4.23}$$

As it turns out:

$$\begin{aligned}
\psi_k^{1,2} &= \frac{1}{\sqrt{2}} \left(\vartheta_k^{1,2} + \overline{\vartheta}_k^{1,2} \right), & \psi_k^{3,4} &= \frac{i}{\sqrt{2}} \left(\vartheta_k^{1,2} - \overline{\vartheta}_k^{1,2} \right), \\
{}_k\psi^{1,2} &= \frac{1}{\sqrt{2}} \left(\vartheta_{k+1}^{1,2} + \overline{\vartheta}_k^{1,2} \right), & {}_k\psi^{3,4} &= \frac{i}{\sqrt{2}} \left(\vartheta_{k+1}^{1,2} - \overline{\vartheta}_k^{1,2} \right).
\end{aligned} \tag{4.24}$$

Consequently, for any polynomial $G((\widetilde{\gamma}^\mu \widetilde{\gamma}^\nu)_{\mu,\nu})$ of order at most 1 in the products $\widetilde{\gamma}^\mu \widetilde{\gamma}^\nu$,

$$\begin{aligned}
\langle \boldsymbol{\vartheta}_{k+1}|\, G((\widetilde{\gamma}_\mu \widetilde{\gamma}_\nu)_{\mu,\nu})\, |\boldsymbol{\vartheta}_k\rangle &= -i \int d^2\overline{\boldsymbol{\vartheta}}_k\, \langle \boldsymbol{\vartheta}_{k+1}\, \big|\overline{\boldsymbol{\vartheta}}_k\big\rangle\, \big\langle \overline{\boldsymbol{\vartheta}}_k\; |\boldsymbol{\vartheta}_k\rangle \cdot G((2 \cdot {}_k\psi^\mu \cdot \psi_k^\nu)_{\mu,\nu}) \\
&= -i \int d^2\overline{\boldsymbol{\vartheta}}_k\;\, e^{(\boldsymbol{\vartheta}_{k+1} - \boldsymbol{\vartheta}_k)\cdot \overline{\boldsymbol{\vartheta}}_k} \cdot G((2 \cdot {}_k\psi^\mu \cdot \psi_k^\nu)_{\mu,\nu})
\end{aligned} \tag{4.25}$$

holds. Thus

$$\langle \mathbf{x}_{k+1}, \boldsymbol{\vartheta}_{k+1}|\, e^{-\varepsilon\mathfrak{D}}\, |\mathbf{x}_k, \boldsymbol{\vartheta}_k\rangle = \langle \mathbf{x}_{k+1}, \boldsymbol{\vartheta}_{k+1}|\, \mathbb{1} - \varepsilon\mathfrak{D} + \mathcal{O}(\varepsilon^2)\, |\mathbf{x}_k, \boldsymbol{\vartheta}_k\rangle$$

$$= \int \frac{d^4\mathbf{p}_k}{(2\pi)^{\frac{4}{2}}}\ \langle \mathbf{x}_{k+1}, \boldsymbol{\vartheta}_{k+1}|\, \mathbb{1} - \varepsilon\mathfrak{D}\left[\hat{\mathbf{p}}, \hat{\mathbf{A}}, (\widetilde{\gamma}_\mu\widetilde{\gamma}_\nu)\right] + \mathcal{O}(\varepsilon^2)\, |\mathbf{p}_k, \boldsymbol{\vartheta}_k\rangle\, e^{-i\mathbf{p}_k\mathbf{x}_k}$$

$$= \int \frac{d^4\mathbf{p}_k}{(2\pi)^{\frac{4}{2}}}\ \langle \boldsymbol{\vartheta}_{k+1}|\, \mathbb{1} - \varepsilon\mathfrak{D}\left[\mathbf{p}_k, \mathbf{A}(\mathbf{x}_{k+1}), (\widetilde{\gamma}_\mu\widetilde{\gamma}_\nu)\right] + \mathcal{O}(\varepsilon^2)\, |\boldsymbol{\vartheta}_k\rangle \cdot \underbrace{\langle \mathbf{x}_{k+1}\, |\mathbf{p}_k\rangle}_{\exp[i\mathbf{p}_k\mathbf{x}_{k+1}]/(2\pi)^{\frac{4}{2}}} \cdot e^{-i\mathbf{p}_k\mathbf{x}_k}$$

$$\overset{(4.25)}{=} \frac{-i}{(2\pi)^4}\int d^4\mathbf{p}_k\ d^2\overline{\boldsymbol{\vartheta}}_k\ e^{(\mathbf{x}_{k+1}-\mathbf{x}_k)\mathbf{p}_k + (\boldsymbol{\vartheta}_{k+1}-\boldsymbol{\vartheta}_k)\cdot\overline{\boldsymbol{\vartheta}}_k}$$

$$\times \left[\mathbb{1} - \varepsilon\mathfrak{D}\left[\mathbf{p}_k, \mathbf{A}(\mathbf{x}_{k+1}), (2\cdot {}_k\psi^\mu\psi^\nu_k)\right] + \mathcal{O}(\varepsilon^2)\right] \quad . \tag{4.26}$$

From (4.21) and (4.26) now follows

$$\mathrm{Tr}\, e^{-T\mathfrak{D}} = i^N \int \prod_{k=1}^{N} d^4\mathbf{x}_k\ d^2\boldsymbol{\vartheta}_k\ \langle \mathbf{x}_{k+1}, \boldsymbol{\vartheta}_{k+1}|\, e^{-\frac{T}{N}\mathfrak{D}}\, |\mathbf{x}_k, \boldsymbol{\vartheta}_k\rangle \qquad \Big| \qquad \mathbf{x}_{N+1} := \mathbf{x}_1,\ \boldsymbol{\vartheta}_{N+1} := -\boldsymbol{\vartheta}_1$$

$$\overset{(4.26)}{=} \frac{(-i)^N i^N}{(2\pi)^{4N}} \int \prod_{k=1}^{N} d^4\mathbf{x}_k\ d^4\mathbf{p}_k\ d^2\boldsymbol{\vartheta}_k\ d^2\overline{\boldsymbol{\vartheta}}_k\ \left[1 - \frac{T}{N}\cdot\mathfrak{D}\left[\mathbf{p}_k, \mathbf{A}[\mathbf{x}_{k+1}], 2\cdot {}_k\psi^\mu\psi^\nu_k\right] + \mathcal{O}\left(\tfrac{T^2}{N^2}\right)\right]$$

$$\times \exp\left[\sum_{k=1}^{N} i(\mathbf{x}_{k+1}-\mathbf{x}_k)\mathbf{p}_k + (\boldsymbol{\vartheta}_{k+1}-\boldsymbol{\vartheta}_k)\cdot\overline{\boldsymbol{\vartheta}}_k\right]$$

$$= \int \prod_{k=1}^{N} \frac{d^4\mathbf{p}_k}{(2\pi)^4}\ d^4\mathbf{x}_k\ d^2\boldsymbol{\vartheta}_k\ d^2\overline{\boldsymbol{\vartheta}}_k\ \left[1 - \frac{T}{N}\cdot\mathfrak{D}\left[\mathbf{p}_k, \mathbf{A}[\mathbf{x}_{k+1}], 2\cdot {}_k\psi^\mu\psi^\nu_k\right] + \mathcal{O}\left(\tfrac{T^2}{N^2}\right)\right]$$

$$\times \exp\left[\sum_{k=1}^{N} i(\mathbf{x}_{k+1}-\mathbf{x}_k)\mathbf{p}_k + \frac{1}{2}(\boldsymbol{\vartheta}_{k+1}-\boldsymbol{\vartheta}_k)\cdot\overline{\boldsymbol{\vartheta}}_k - \frac{1}{2}\boldsymbol{\vartheta}_k\cdot(\overline{\boldsymbol{\vartheta}}_k - \overline{\boldsymbol{\vartheta}}_{k-1})\right] \qquad \Big| \qquad \overline{\boldsymbol{\vartheta}}_0 := -\overline{\boldsymbol{\vartheta}}_N \quad .$$

$$\tag{4.27}$$

We identify $\mathbf{x}_k$, $\mathbf{p}_k$ and $\boldsymbol{\vartheta}_k, \overline{\boldsymbol{\vartheta}}_k$ as coordinates of a trajectory defined at *times* $\tau_k := k\cdot\varepsilon$, whereas $\varepsilon := T/N$. Then $(\mathbf{x}_{k+1}-\mathbf{x}_k)/\varepsilon$, $(\boldsymbol{\vartheta}_{k+1}-\boldsymbol{\vartheta}_k)/\varepsilon$ and $(\overline{\boldsymbol{\vartheta}}_k - \overline{\boldsymbol{\vartheta}}_{k-1})/\varepsilon$ go for $N\to\infty$ over to the velocities $\dot{\mathbf{x}}$, $\dot{\boldsymbol{\vartheta}}$ and $\dot{\overline{\boldsymbol{\vartheta}}}$. Furthermore, the sum in (4.27) goes over to a Riemannian one, yielding the path-integral representation

$$\mathrm{Tr}\, e^{-T\mathfrak{D}} = \oint D\mathbf{x}\ D\mathbf{p}\ D\boldsymbol{\vartheta}\ D\overline{\boldsymbol{\vartheta}}\ \exp\left[\int_0^T \left[i\dot{\mathbf{x}}\mathbf{p} + \frac{1}{2}\dot{\boldsymbol{\vartheta}}\cdot\overline{\boldsymbol{\vartheta}} - \frac{1}{2}\boldsymbol{\vartheta}\cdot\dot{\overline{\boldsymbol{\vartheta}}} - \mathfrak{D}\left[\mathbf{p}, \mathbf{A}, 2\psi^\mu\psi^\nu\right]\right] d\tau\right], \tag{4.28}$$

with boundary conditions $\boldsymbol{\vartheta}(T) = -\boldsymbol{\vartheta}(0)$, $\overline{\boldsymbol{\vartheta}}(T) = -\overline{\boldsymbol{\vartheta}}(0)$ and $\mathbf{x}(T) = \mathbf{x}(0)$. Here, the variables ${}_k\psi^\mu$, ψ^μ_k turn to

$$\psi^{1,2}(\tau) = \frac{1}{\sqrt{2}}\left[\vartheta^{1,2}(\tau) + \overline{\vartheta}^{1,2}(\tau)\right] \quad , \quad \psi^{3,0}(\tau) = \frac{i}{\sqrt{2}}\left[\vartheta^{1,2}(\tau) - \overline{\vartheta}^{1,2}(\tau)\right], \tag{4.29}$$

with ψ^μ_k, ${}_k\psi^\mu \to \psi^\mu(t_k)$ (compare to (4.24)). In particular

$$\dot{\boldsymbol{\vartheta}}\cdot\overline{\boldsymbol{\vartheta}} - \boldsymbol{\vartheta}\cdot\dot{\overline{\boldsymbol{\vartheta}}} = -\boldsymbol{\psi}\dot{\boldsymbol{\psi}} \quad . \tag{4.30}$$

Thus, (4.28) takes the form[29]

$$\mathrm{Tr}\, e^{-T\mathfrak{D}} = \oint\limits_P D\mathbf{x}\, D\mathbf{p} \oint\limits_A D\boldsymbol{\psi}\ \exp\left[\int\limits_0^T \left[i\cdot\dot{\mathbf{x}}\mathbf{p} - \frac{1}{2}\boldsymbol{\psi}\dot{\boldsymbol{\psi}} - \mathfrak{D}\left[\mathbf{p}, \mathbf{A}, (2\psi^\mu\psi^\nu)\right]\right] d\tau\right], \tag{4.31}$$

with boundary conditions $\boldsymbol{\psi}(T) = -\boldsymbol{\psi}(0)$ and $\mathbf{x}(T) = \mathbf{x}(0)$. Here, "$\oint\limits_P$" and "$\oint\limits_A$" denote *periodic* and *anti-periodic* boundary conditions. Since $\mathfrak{D}[\mathbf{p}, \mathbf{A}, (2\psi^\mu\psi^\nu)]$ is quadratic in $\mathbf{p}$, the momentum-integral can formally be evaluated as a Gaussian one to yield

$$\mathrm{Tr}\, e^{-T\mathfrak{D}} = \oint\limits_P D\mathbf{x} \oint\limits_A D\boldsymbol{\psi}\ \exp\left[-\int\limits_0^T \mathcal{L}_{\mathrm{sp}}(\mathbf{x}, \dot{\mathbf{x}}, \boldsymbol{\psi}, \dot{\boldsymbol{\psi}})\ d\tau\right], \tag{4.32}$$

with

$$\boxed{\mathcal{L}_{\mathrm{sp}}(\mathbf{x}, \dot{\mathbf{x}}, \psi, \dot{\psi}) := \frac{\dot{\mathbf{x}}^2}{4} + \frac{1}{2}\boldsymbol{\psi}\dot{\boldsymbol{\psi}} - ie\dot{\mathbf{x}}\mathbf{A} + ie\psi^\mu\psi^\nu F_{\mu\nu} + m^2.} \tag{4.33}$$

Finally, from (4.32) follows for (4.12) the representation

$$\boxed{\Gamma[\mathbf{A}] = \frac{i}{2}\int\limits_0^\infty \frac{dT}{T} \oint\limits_P D\mathbf{x} \oint\limits_A D\boldsymbol{\psi}\ e^{-\int_0^T d\tau\ \mathcal{L}_{\mathrm{sp}}(\mathbf{x}, \dot{\mathbf{x}}, \boldsymbol{\psi}, \dot{\boldsymbol{\psi}})}} \tag{4.34}$$

for the one-loop correction to the effective action, as introduced in section 4.1. Note that ψ^μ are actually Grassmann-variables and $\boldsymbol{\psi}$ **not** a 4-vector. The integral $\oint\limits_A D\boldsymbol{\psi}$ is a Grassman path integral.

Also note that the periodic boundary condition on the paths $\mathbf{x}(\tau)$ secures the gauge invariance of the effective action, since for any gauge transformation $A_\mu \to A_\mu + \partial_\mu f$, the exponent in (4.34) only changes by a term

$$\int\limits_0^T d\tau\ ie\dot{x}^\mu \partial_\mu f = ie\int\limits_0^T d\tau\ \frac{d}{d\tau} f(\mathbf{x}(\tau)) = ie\left[V(\mathbf{x}(T)) - V(\mathbf{x}(0))\right] = 0. \tag{4.35}$$

In the following, we shall assume a spacetime dimension D, preparing for dimensional regularization[16].

4.3 N-point functions

Starting from result (4.34), similarly to section 3.3.2, we obtain the Fourier-transform

$$\widetilde{\Gamma}_N[\mathbf{p}^1, .., \mathbf{p}^N; \boldsymbol{\varepsilon}^1, .., \boldsymbol{\varepsilon}^N] = \frac{i}{2}\frac{(ie)^N}{(2\pi)^{DN/2}} \int\limits_0^\infty \frac{dT}{T} e^{-m^2 T} \oint\limits_P D\mathbf{x}\ e^{-\int_0^T \frac{\dot{\mathbf{x}}^2}{4} d\tau} \oint\limits_A D\boldsymbol{\psi}\ e^{-\int_0^T \frac{d\tau}{2}\boldsymbol{\psi}\dot{\boldsymbol{\psi}}}$$

$$\times \prod_{k=1}^N \int\limits_0^T d\tau_k \left[\boldsymbol{\varepsilon}^k \underbrace{\dot{\mathbf{x}}(\tau_k)}_{=:\mathbf{x}_k} + 2i(\boldsymbol{\varepsilon}^k \underbrace{\boldsymbol{\psi}(\tau_k)}_{=:\boldsymbol{\psi}_k})(\mathbf{p}^k\boldsymbol{\psi}(\tau_k))\right] \cdot e^{i\mathbf{p}^k\mathbf{x}(\tau_k)} \tag{4.36}$$

for the one-loop contribution to the N-point photon correlator[30]. Introducing the Grassmann variables $\vartheta_k, \overline{\vartheta}_k$, we can write[31]

$$\boldsymbol{\varepsilon}^k\dot{\mathbf{x}}_k + 2i(\boldsymbol{\varepsilon}^k\boldsymbol{\psi}_k)(\mathbf{p}^k\boldsymbol{\psi}_k) = \int d\overline{\vartheta}_k d\vartheta_k\ \exp\left[\overline{\vartheta}_k\vartheta_k(\boldsymbol{\varepsilon}^k\dot{\mathbf{x}}_k) + \sqrt{2}\vartheta_k(\boldsymbol{\varepsilon}^k\boldsymbol{\psi}_k) + \sqrt{2}i\overline{\vartheta}_k(\mathbf{p}^k\boldsymbol{\psi}_k)\right]. \tag{4.37}$$

[29] The Jacobian det $\frac{\partial\psi}{\partial(\boldsymbol{\vartheta}, \overline{\boldsymbol{\vartheta}})} = -1$ shall be absorbed into the functional measure.

[30] Defined as $\frac{\delta^N\Gamma}{\delta\mathbf{A}(\mathbf{x}_1)..\delta\mathbf{A}(\mathbf{x}_N)}\Big|_{\mathbf{A}=0}$.

[31] Using the convention $\int d\overline{\vartheta}_k\ d\vartheta_k\ \overline{\vartheta}_k\vartheta_k = 1$.

Using the same method as in (3.2.2), we separate the integral $\oint D\mathbf{x}$ to $\int d\overline{\mathbf{x}} \oint\limits_{\overline{\mathbf{y}}=0} D\mathbf{y}$, so that (4.36) takes the form

$$\widetilde{\Gamma}_N[\mathbf{p}^1,..,\mathbf{p}^N;\boldsymbol{\varepsilon}^1,..,\boldsymbol{\varepsilon}^N] = \frac{i}{2}\frac{(ie)^N}{(2\pi)^{DN/2}}\int\limits_0^\infty \frac{dT}{T}e^{-m^2T}\underbrace{\int d\overline{\mathbf{x}}\ e^{i\sum_{k=1}^N \mathbf{p}^k\overline{\mathbf{x}}}}_{(2\pi)^D\delta\left[\sum_{k=1}^N \mathbf{p}^k\right]}\oint\limits_{\overline{\mathbf{y}}=0} D\mathbf{y}\ e^{-\int\limits_0^T \frac{\dot{\mathbf{y}}^2}{4}d\tau}\oint\limits_A D\boldsymbol{\psi}\ e^{-\int\limits_0^T \frac{d\tau}{2}\boldsymbol{\psi}\dot{\boldsymbol{\psi}}}$$

$$\times\prod_{k=1}^N\int\limits_0^T d\tau_k\int d\overline{\vartheta}_k d\vartheta_k\ e^{\sqrt{2}\vartheta_k(\boldsymbol{\varepsilon}^k\boldsymbol{\psi}_k)+\sqrt{2}i\overline{\vartheta}_k(\mathbf{p}^k\boldsymbol{\psi}_k)}\cdot e^{\overline{\vartheta}_k\vartheta_k(\boldsymbol{\varepsilon}^k\dot{\mathbf{y}}_k)+i\mathbf{p}^k\mathbf{y}(\tau_k)}$$

$$=\frac{i}{2}\frac{(-ie)^N(2\pi)^D}{(2\pi)^{DN/2}}\cdot\delta\left[\sum_{k=1}^N\mathbf{p}^k\right]\cdot\int\limits_0^\infty\frac{dT}{T}e^{-m^2T}\prod_{k=1}^N\left[\int\limits_0^T d\tau_k\ \int d\overline{\vartheta}_k d\vartheta_k\right]$$

$$\oint\limits_{\overline{\mathbf{y}}=0} D\mathbf{y}\ e^{-\left\langle\mathbf{y},-\frac{1}{4}\frac{d^2}{d\tau^2}\mathbf{y}\right\rangle_T+\langle\mathbf{J},\mathbf{y}\rangle_T}\oint\limits_A D\mathbf{y}\ e^{-\left\langle\boldsymbol{\psi},\frac{1}{2}\frac{d}{d\tau}\boldsymbol{\psi}\right\rangle_T+\langle\boldsymbol{\eta},\boldsymbol{\psi}\rangle_T}, \tag{4.38}$$

with the dual forms

$$\mathbf{J}(\vartheta_k,\overline{\vartheta}_k) = \sum_{k=1}^N\delta(\tau_k-\cdot)\cdot\left[\overline{\vartheta}_k\vartheta_k\boldsymbol{\varepsilon}^k\partial_\tau+i\mathbf{p}^k\right]$$

$$\boldsymbol{\eta}(\vartheta_k,\overline{\vartheta}_k) = \sqrt{2}\cdot\sum_{k=1}^N\delta(\tau_k-\cdot)\cdot\left[\vartheta_k\boldsymbol{\varepsilon}^k+i\overline{\vartheta}_k\mathbf{p}^k\right]\quad . \tag{4.39}$$

Using the conventions

$$\oint\limits_{\overline{\mathbf{y}}=0} D\mathbf{y}\ e^{-\left\langle\mathbf{y},-\frac{1}{4}\frac{d^2}{d\tau^2}\mathbf{y}\right\rangle_T} = (4\pi T)^{-\frac{D}{2}}$$

$$\oint\limits_A D\boldsymbol{\psi}\ e^{-\left\langle\boldsymbol{\psi},\frac{1}{2}\frac{d}{d\tau}\boldsymbol{\psi}\right\rangle_T} = 4 \tag{4.40}$$

we can evaluate the path-integrals in (4.38) as Gaussian ones and obtain[32]

$$
\begin{aligned}
\widetilde{\Gamma}_N[\mathbf{p}^1,..,\mathbf{p}^N;\boldsymbol{\varepsilon}^1,..,\boldsymbol{\varepsilon}^N] =& i\frac{2\pi^{\frac{D}{2}}(ie)^N}{(2\pi)^{DN/2}}\cdot\delta\left[\sum_{k=1}^N \mathbf{p}^k\right]\cdot\int_0^\infty \frac{dT}{T^{1+\frac{D}{2}}}e^{-m^2T}\prod_{k=1}^N\left[\int_0^T d\tau_k \int d\overline{\vartheta}_k d\vartheta_k\right] \\
& \exp\left[\left\langle \mathbf{J},\left(-\tfrac{d^2}{d\tau^2}\right)^{-1}\mathbf{J}\right\rangle_T\right]\cdot\exp\left[-\frac{1}{2}\left\langle \boldsymbol{\eta},\left(\tfrac{d}{d\tau}\right)^{-1}\boldsymbol{\eta}\right\rangle_T\right] \\
=& i\frac{2\pi^{\frac{D}{2}}(ie)^N}{(2\pi)^{DN/2}}\cdot\delta\left[\sum_{k=1}^N \mathbf{p}^k\right]\cdot\int_0^\infty \frac{dT}{T^{1+\frac{D}{2}}}e^{-m^2T}\prod_{k=1}^N\left[\int_0^T d\tau_k \int d\overline{\vartheta}_k d\vartheta_k\right] \\
& \exp\left[-\sum_{k,l=1}^N\left[\overline{\vartheta}_k\vartheta_k\boldsymbol{\varepsilon}^k\partial_1+i\mathbf{p}^k\right]\left[\overline{\vartheta}_l\vartheta_l\boldsymbol{\varepsilon}^l\partial_2+i\mathbf{p}^l\right]\underbrace{\left\langle\delta(\tau_k-\cdot),\left(\tfrac{d^2}{d\tau^2}\right)^{-1}\delta(\tau_l-\cdot)\right\rangle}_{\frac{1}{2}G_B(\tau_k,\tau_l)}\right] \\
& \times\exp\left[-\sum_{k,l=1}^N\left[\vartheta_k\boldsymbol{\varepsilon}^k+i\overline{\vartheta}_k\mathbf{p}^k\right]\left[\vartheta_l\boldsymbol{\varepsilon}^l+i\overline{\vartheta}_l\mathbf{p}^l\right]\underbrace{\left\langle\delta(\tau_k-\cdot),\left(\tfrac{d}{d\tau}\right)^{-1}\delta(\tau_l-\cdot)\right\rangle}_{\frac{1}{2}G_F(\tau_k,\tau_l)}\right] \\
=& i\frac{2\pi^{\frac{D}{2}}(ie)^N}{(2\pi)^{DN/2}}\cdot\delta\left[\sum_{k=1}^N \mathbf{p}^k\right]\cdot\int_0^\infty \frac{dT}{T^{1+\frac{D}{2}}}e^{-m^2T}\prod_{k=1}^N\left[\int_0^T d\tau_k \int d\overline{\vartheta}_k d\vartheta_k\right] \\
& \exp\left[\frac{1}{2}\sum_{k,l=1}^N\left[\mathbf{p}^k\mathbf{p}^l-\overline{\vartheta}_k\vartheta_k\overline{\vartheta}_l\vartheta_l\boldsymbol{\varepsilon}^k\boldsymbol{\varepsilon}^l\partial^2_{1,2}-i\overline{\vartheta}_k\vartheta_k\boldsymbol{\varepsilon}^k\mathbf{p}^l\partial_1-i\overline{\vartheta}_l\vartheta_l\boldsymbol{\varepsilon}^l\mathbf{p}^k\partial_2\right]G_B(\tau_k,\tau_l)\right] \\
& \times\exp\left[\frac{1}{2}\sum_{k,l=1}^N\left[\overline{\vartheta}_k\overline{\vartheta}_l\mathbf{p}^k\mathbf{p}^l-\vartheta_k\vartheta_l\boldsymbol{\varepsilon}^k\boldsymbol{\varepsilon}^l-i\vartheta_k\overline{\vartheta}_l\boldsymbol{\varepsilon}^k\mathbf{p}^l-i\overline{\vartheta}_k\vartheta_l\mathbf{p}^k\boldsymbol{\varepsilon}^l\right]G_F(\tau_k,\tau_l)\right] \\
=& i\frac{2\pi^{\frac{D}{2}}(ie)^N}{(2\pi)^{DN/2}}\cdot\delta\left[\sum_{k=1}^N \mathbf{p}^k\right]\cdot\int_0^\infty \frac{dT}{T^{1+\frac{D}{2}}}e^{-m^2T}\prod_{k=1}^N\left[\int_0^T d\tau_k \int d\overline{\vartheta}_k d\vartheta_k\right] \\
& \exp\left[\frac{1}{2}\sum_{k,l=1}^N\left[\mathbf{p}^k\mathbf{p}^l-2i\overline{\vartheta}_l\vartheta_l\boldsymbol{\varepsilon}^l\mathbf{p}^k\partial_1+\overline{\vartheta}_k\vartheta_k\overline{\vartheta}_l\vartheta_l\boldsymbol{\varepsilon}^k\boldsymbol{\varepsilon}^l\partial_1^2\right]G_B(\tau_k,\tau_l)\right] \\
& \times\exp\left[\frac{1}{2}\sum_{k,l=1}^N\left[\overline{\vartheta}_k\overline{\vartheta}_l\mathbf{p}^k\mathbf{p}^l-\vartheta_k\vartheta_l\boldsymbol{\varepsilon}^k\boldsymbol{\varepsilon}^l-2i\overline{\vartheta}_k\vartheta_l\mathbf{p}^k\boldsymbol{\varepsilon}^l\right]G_F(\tau_k,\tau_l)\right],
\end{aligned}
\tag{4.41}
$$

whereas in the last step we used (3.44) and $G_F(x,y)=\mathrm{sgn}(x-y)$.

[32]See Swanson[1] about Gaussian Grassmann integrals.

4.4 The vacuum-polarization tensor

For $N = 2$, (4.41) takes the form

$$\widetilde{\Gamma}_2[\mathbf{p}^1, \mathbf{p}^2; \boldsymbol{\varepsilon}^1, \boldsymbol{\varepsilon}^2] = i\frac{2\pi^{\frac{D}{2}}(ie)^2}{(2\pi)^D} \cdot \delta\left[\mathbf{p}^1 + \mathbf{p}^2\right] \cdot \int_0^\infty \frac{dT}{T^{1+\frac{D}{2}}} e^{-m^2 T} \int_0^T d\tau_1 d\tau_2 \int d\overline{\vartheta}_1 d\vartheta_1 d\overline{\vartheta}_2 d\vartheta_2$$

$$\exp\left[\left(\mathbf{p}^1\mathbf{p}^2 + i\overline{\vartheta}_1\vartheta_1\boldsymbol{\varepsilon}^1\mathbf{p}^2\partial_1 - i\overline{\vartheta}_2\vartheta_2\boldsymbol{\varepsilon}^2\mathbf{p}^1\partial_1 + \overline{\vartheta}_1\vartheta_1\overline{\vartheta}_2\vartheta_2\boldsymbol{\varepsilon}^1\boldsymbol{\varepsilon}^2\partial_1^2\right) G_B(\tau_1, \tau_2)\right]$$

$$\times \exp\left[\left(\overline{\vartheta}_1\overline{\vartheta}_2\mathbf{p}^1\mathbf{p}^2 - \vartheta_1\vartheta_2\boldsymbol{\varepsilon}^1\boldsymbol{\varepsilon}^2 - i\overline{\vartheta}_1\vartheta_2\mathbf{p}^1\boldsymbol{\varepsilon}^2 + i\overline{\vartheta}_2\vartheta_1\mathbf{p}^2\boldsymbol{\varepsilon}^1\right) G_F(\tau_1, \tau_2)\right]$$

$$= -i\frac{2\pi^{\frac{D}{2}}e^2}{(2\pi)^D} \cdot \delta\left[\mathbf{p}^1 + \mathbf{p}^2\right] \cdot \int_0^\infty \frac{dT}{T^{\frac{D}{2}}} e^{-m^2 T} \int_0^T d\tau\ e^{\mathbf{p}^1\mathbf{p}^2 G_B(\tau)}$$

$$\left[(\boldsymbol{\varepsilon}^1\boldsymbol{\varepsilon}^2)\ddot{G}_B(\tau) + (\mathbf{p}^1\boldsymbol{\varepsilon}^2)(\mathbf{p}^2\boldsymbol{\varepsilon}^1)\big(\dot{G}_B(\tau)\big)^2 + \left[(\mathbf{p}^1\mathbf{p}^2)(\boldsymbol{\varepsilon}^1\boldsymbol{\varepsilon}^2) - (\mathbf{p}^1\boldsymbol{\varepsilon}^2)(\mathbf{p}^2\boldsymbol{\varepsilon}^1)\right] G_F^2(\tau)\right]$$

$$= -i\frac{2\pi^{\frac{D}{2}}e^2}{(2\pi)^D} \cdot \delta\left[\mathbf{p}^1 + \mathbf{p}^2\right] \cdot \int_0^\infty \frac{dT}{T^{\frac{D}{2}}} e^{-m^2 T} \int_0^T d\tau\ e^{\mathbf{p}^1\mathbf{p}^2 G_B(\tau)}$$

$$\left[(\mathbf{p}^1\mathbf{p}^2)(\boldsymbol{\varepsilon}^1\boldsymbol{\varepsilon}^2) - (\mathbf{p}^1\boldsymbol{\varepsilon}^2)(\mathbf{p}^2\boldsymbol{\varepsilon}^1)\right] \cdot \left[G_F^2(\tau) - \big(\dot{G}_B(\tau)\big)^2\right], \tag{4.42}$$

whereas in the last step a partial integration of $\int_0^T d\tau\ e^{\mathbf{p}^1\mathbf{p}^2 G_B(\tau)}\ddot{G}_B(\tau)$ was performed. In analogy to section 3.3.2, the vacuum polarization tensor is thus given by

$$\widetilde{\Pi}^{\mu\nu}[\mathbf{p}] = -\frac{i}{(2\pi)^{\frac{D}{2}}}\frac{2\pi^{\frac{D}{2}}e^2}{(2\pi)^D} \cdot \left[p_\mu p_\nu - (\mathbf{p})^2\delta_{\mu\nu}\right] \cdot \int_0^\infty \frac{dT}{T^{\frac{D}{2}}} e^{-m^2 T} \int_0^T d\tau\ e^{-(\mathbf{p})^2 G_B(\tau)} \cdot \left[G_F^2(\tau) - \big(\dot{G}_B(\tau)\big)^2\right]$$

$$\overset{u:=\frac{\tau}{T}}{=} -\frac{i}{(2\pi)^{\frac{D}{2}}}\frac{2\pi^{\frac{D}{2}}e^2}{(2\pi)^D} \cdot \left[p_\mu p_\nu - (\mathbf{p})^2\delta_{\mu\nu}\right] \cdot \int_0^\infty \frac{dT}{T^{\frac{D}{2}-1}} e^{-m^2 T} \int_0^1 du\ e^{-(\mathbf{p})^2 uT(1-u)} \cdot \underbrace{\left[1 - (1-2u)^2\right]}_{4u(1-u)}$$

$$= -\frac{i}{(2\pi)^{\frac{D}{2}}}\frac{8e^2}{(4\pi)^{\frac{D}{2}}} \cdot \left[p_\mu p_\nu - (\mathbf{p})^2\delta_{\mu\nu}\right] \cdot \Gamma\left(2 - \tfrac{D}{2}\right) \cdot \int_0^1 du\ u(1-u)\left[m^2 + (\mathbf{p})^2 u(1-u)\right]^{\frac{D}{2}-2}. \tag{4.43}$$

Note: Note that the results in sections 4.3 & 4.4 for the 1PI-N-point functions and polarization tensor, are all in Euclidean form. The return to Minkowskian coordinates is performed as described in appendix A.5. Thus for example, the Minkowskian polarization tensor $\Pi_M^{\mu\nu}$ takes the form

$$\widetilde{\Pi}_M^{\mu\nu}[\mathbf{p}] = \frac{1}{(2\pi)^{\frac{D}{2}}}\frac{8e^2}{(4\pi)^{\frac{D}{2}}} \cdot \left[p^\mu p^\nu - g(\mathbf{p},\mathbf{p}) \cdot g^{\mu\nu}\right] \cdot \Gamma\left(2 - \tfrac{D}{2}\right) \cdot \int_0^1 du\ u(1-u)\left[m^2 - g(\mathbf{p},\mathbf{p})u(1-u)\right]^{\frac{D}{2}-2}. \tag{4.44}$$

which is, overseeing the incomplete regularization, in accordance[33] with Greiner[22], .

[33] Up to the factor $\frac{1}{(2\pi)^{\frac{D}{2}}}$, which is due to our use of isometrical Fourier-transforms.

A Appendix

This appendix shall provide with some auxiliary statements, used along the article. Most of them can also be found in standard literature in this or slightly modified form.

A.1 A note on Dirac matrices

Let $\zeta^1, .., \zeta^n$ be complex $n \times n$ matrices satisfying $\{\zeta^\mu, \zeta^\nu\} = 2g^{\mu\nu}$ for some $n \times n$ matrix g. Let $\mathbf{A}$ be some differentiable field and $\hat{p}_\mu := -i\partial_\mu$. Then

$$(\zeta^\mu \hat{p}_\mu + \zeta^\mu e A_\mu)^2 = g^{\mu\nu}(\hat{p}_\mu + eA_\mu)(\hat{p}_\nu + eA_\nu) \cdot \mathbb{1} - \frac{i}{4} e \left[\zeta^\mu, \zeta^\nu\right] F_{\mu\nu} \tag{A.1}$$

holds, whereas $F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$.

Proof

We note that

$$\left[\zeta^\mu, \zeta^\nu\right] F_{\mu\nu} = \left[\zeta^\mu, \zeta^\nu\right] (\partial_\mu A_\nu - \partial_\nu A_\mu) = 2 \left[\zeta^\mu, \zeta^\nu\right] \partial_\mu A_\nu = 2i \left[\zeta^\mu, \zeta^\nu\right] \hat{p}_\mu A_\nu \tag{A.2}$$

and obtain

$$\begin{aligned}
(\zeta^\mu \hat{p}_\mu + \zeta^\mu e A_\mu)^2 &= \zeta^\mu \zeta^\nu \left[\hat{p}_\mu + eA_\mu\right] \left[\hat{p}_\nu + eA_\nu\right] \\
&= \underbrace{\zeta^\mu \zeta^\nu \hat{p}_\mu \hat{p}_\nu}_{\frac{1}{2}\{\zeta^\mu,\zeta^\nu\}\hat{p}_\mu\hat{p}_\nu} + e^2 \underbrace{\zeta^\mu \zeta^\nu A_\mu A_\nu}_{\frac{1}{2}\{\zeta^\mu,\zeta^\nu\}A_\mu A_\nu} + e\zeta^\mu\zeta^\nu \Big[\underbrace{\hat{p}_\mu (A_\nu \cdot)}_{(\hat{p}_\mu A_\nu) + A_\nu \hat{p}_\mu} + A_\mu \hat{p}_\nu \Big] \\
&= g^{\mu\nu} p_\mu p_\nu + e^2 g^{\mu\nu} A_\mu A_\nu + e \underbrace{\{\zeta^\mu, \zeta^\nu\}}_{2g^{\mu\nu}} A_\mu \hat{p}_\nu + e \underbrace{\zeta^\mu \zeta^\nu}_{\frac{1}{2}[[\zeta^\mu,\zeta^\nu] + \{\zeta^\mu,\zeta^\nu\}]} (\hat{p}_\mu A_\nu) \\
&= g^{\mu\nu} \hat{p}_\mu \hat{p}_\nu + e^2 g^{\mu\nu} A_\mu A_\nu + 2eg^{\mu\nu} A_\mu \hat{p}_\nu + \frac{e}{2} \underbrace{\{\zeta^\mu, \zeta^\nu\}}_{2g^{\mu\nu}} (\hat{p}_\mu A_\nu) + \frac{e}{2} \left[\zeta^\mu, \zeta^\nu\right] (\hat{p}_\mu A_\nu) \\
&= g^{\mu\nu} (\hat{p}_\mu + eA_\mu)(\hat{p}_\nu + eA_\nu) + \underbrace{\frac{e}{2} \left[\zeta^\mu, \zeta^\nu\right] (\hat{p}_\mu A_\nu)}_{-\frac{ie}{4}[\zeta^\mu,\zeta^\nu]F_{\mu\nu} \text{ by (A.2)}}
\end{aligned} \tag{A.3}$$

as claimed.
□

A.2 Differential operators on compact sets

Consider the differential operator $\frac{d^2}{d\tau^2}$ on the space $\mathcal{P}[0, T]$ of integrable, periodic functions on the interval $[0, T]$. Call two functions $f, g \in \mathcal{F}[0, T]$ equivalent $f \sim g$, if $f - g = \text{const}$ and consider the family $\mathcal{F}[0, T]/\sim$ of equivalence classes $[f]_\sim$ on the resulting quotient space. Then $f'' = g''$ iff $[f]_\sim = [g]_\sim$, hence $\mathfrak{D}\,[f]_\sim := f''$ is a well-defined, invertible operator on $\mathcal{F}[0, T]/\sim$. Its inverse $I := \mathfrak{D}^{-1}$ is characterized by

$$\left(I\left[\delta(\tau_1 - \cdot)\right]\right)(t) = \left[\frac{|t - \tau_1|}{2} - \frac{(t - \tau_1)^2}{2T}\right]_\sim. \tag{A.4}$$

Indeed, applying $\mathfrak{D}$ to (A.4) results in

$$\frac{d^2}{dt^2}\left(\frac{|t - \tau_1|}{2} - \frac{(t - \tau_1)^2}{2T}\right) = \delta(\tau_1 - t) - \frac{1}{T}. \tag{A.5}$$

In a very similar way, one can show that the operator $\frac{d}{d\tau}$ on the space of antiperiodic functions on $[0,T]$ is invertible, with Green's function

$$\left(\left[\tfrac{d}{d\tau}\right]^{-1}[\delta(\tau_1-\cdot)]\right)(t) = \frac{1}{2}\,\mathrm{sgn}(t-\tau_1) =: \frac{1}{2}G_F(t,\tau_1) \quad . \tag{A.6}$$

A.3 Evaluating quadratic Hamiltonians

Let the Hamiltonian of an n-dimensional system be given by[34]

$$\hat{H}(\hat{\mathbf{x}},\hat{\mathbf{p}},t) = \frac{1}{2}\hat{\mathbf{p}}^T\mathbb{M}^{-1}\hat{\mathbf{p}} + \frac{1}{2}\left\{\mathbf{a}(\hat{\mathbf{x}},t),\hat{\mathbf{p}}\right\} + V(\hat{\mathbf{x}},t) \quad , \tag{A.7}$$

with $\mathbb{M}\in\mathbb{R}^{n\times n}$ as symmetric, invertible matrix and $\mathbf{a}:\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}^n$. Then the transition element is given by[35]

$$\langle\mathbf{x}_\beta|\, e^{-i(t_\beta-t_\alpha)\hat{H}}\,|\mathbf{x}_\alpha\rangle = \int_{(t_\alpha,\mathbf{x}_\alpha)}^{(t_\beta,\mathbf{x}_\beta)} \widetilde{D}\mathbf{x}\; e^{\,i\int_{t_\alpha}^{t_\beta}\mathcal{L}_c(\mathbf{x},\dot{\mathbf{x}},t)\,dt} \quad , \tag{A.8}$$

with

$$\mathcal{L}_c(\mathbf{x},\dot{\mathbf{x}},t) := \frac{1}{2}\dot{\mathbf{x}}^T\mathbb{M}\dot{\mathbf{x}} + \frac{1}{2}\mathbf{a}(\mathbf{x},t)^T\mathbb{M}\mathbf{a}(\mathbf{x},t) - \dot{\mathbf{x}}^T\mathbb{M}\mathbf{a}(\mathbf{x},t) - V(\mathbf{x},t) \tag{A.9}$$

and $\mathcal{L}_c$ as the classical Lagrangian of (A.7).

A.4 On identities, traces and coherent states

Consider the coherent states defined in (4.16), parametrized by the Grassmann variables $\vartheta^1,\vartheta^2,\overline{\vartheta}^1,\overline{\vartheta}^2$. We define the Fock-states

$$|n_1,n_2\rangle := (a_2^+)^{n_2}(a_1^+)^{n_1}\,|0\rangle \quad , \quad n_1,n_2\in\{0,1\} \tag{A.10}$$

forming an orthonormal basis in the Fock-space $\mathcal{H}_a$, generated by the $a^\pm_{1,2}$. We notice that the coherent states can be written as

$$\begin{aligned}|\boldsymbol{\vartheta}\rangle &= |0\rangle - \vartheta^1\,|1,0\rangle - \vartheta^2\,|0,1\rangle + \vartheta^1\vartheta^2\,|1,1\rangle\,,\\ \langle\boldsymbol{\vartheta}| &= i\,\langle 0|\,\vartheta^1\vartheta^2 - i\,\langle 1,0|\,\vartheta^2 + i\,\langle 0,1|\,\vartheta^1 + i\,\langle 1,1|\,,\end{aligned} \tag{A.11}$$

whereas we used that the $a^\pm_{1,2}$ anticommute with $\vartheta^{1,2}$. Using the rules of Grassmann integration we conclude

$$\begin{aligned} i\int d^2\boldsymbol{\vartheta}\;\langle-\boldsymbol{\vartheta}|\,U\,|\boldsymbol{\vartheta}\rangle &\overset{(A.11)}{=} i^2\int d^2\boldsymbol{\vartheta}\left[\underbrace{\langle 0|\,\vartheta^1\vartheta^2U\,|0\rangle}_{\vartheta^1\vartheta^2\langle 0|U|0\rangle} - \underbrace{\langle 1,0|\,\vartheta^2U\vartheta^1\,|1,0\rangle}_{-\vartheta^1\vartheta^2\langle 1,0|U|1,0\rangle} + \underbrace{\langle 0,1|\,\vartheta^1U\vartheta^2\,|0,1\rangle}_{\vartheta^1\vartheta^2\langle 0,1|U|0,1\rangle} + \underbrace{\langle 1,1|\,U\vartheta^1\vartheta^2\,|1,1\rangle}_{\vartheta^1\vartheta^2\langle 1,1|U|1,1\rangle}\right]\\ &= -\underbrace{\int d\vartheta^1 d\vartheta^2\;\vartheta^1\vartheta^2}_{(-i)^2}\;\mathrm{trace}(U) = \mathrm{trace}(U) \quad , \end{aligned} \tag{A.12}$$

that is

$$\boxed{i\int d^2\boldsymbol{\vartheta}\;\langle-\boldsymbol{\vartheta}|\,U\,|\boldsymbol{\vartheta}\rangle = \mathrm{trace}(U).} \tag{A.13}$$

[34] Here, t is some arbitrary parameter of the trajectories.

[35] The exact phase of the measure $\widetilde{D}\mathbf{x}$ depends among others, on the signature of $\mathbb{M}$.

Similarly, from (4.16) one has the representation

$$|\overline{\boldsymbol{\vartheta}}\rangle = i\overline{\vartheta}^1\overline{\vartheta}^2\,|0\rangle + i\overline{\vartheta}^2\,|1,0\rangle - i\overline{\vartheta}^1\,|0,1\rangle - i\,|1,1\rangle$$

$$\langle\overline{\boldsymbol{\vartheta}}| = \langle 0| - \langle 1,0|\,\overline{\vartheta}^1 - \langle 0,1|\,\overline{\vartheta}^2 - \langle 1,1|\,\overline{\vartheta}^1\overline{\vartheta}^2. \tag{A.14}$$

Consequently

$$-i\int d^2\overline{\boldsymbol{\vartheta}}\;\;|\overline{\boldsymbol{\vartheta}}\rangle\langle\overline{\boldsymbol{\vartheta}}| \overset{(A.14)}{=} -i^2\int d^2\overline{\boldsymbol{\vartheta}}\;\Big[\overline{\vartheta}^1\overline{\vartheta}^2\,|0\rangle\langle 0| - \underbrace{\overline{\vartheta}^2\,|1,0\rangle\langle 1,0|\,\overline{\vartheta}^1}_{\overline{\vartheta}^2\overline{\vartheta}^1|1,0\rangle\langle 1,0|} + \underbrace{\overline{\vartheta}^1\,|0,1\rangle\langle 0,1|\,\overline{\vartheta}^2}_{-\overline{\vartheta}^2\overline{\vartheta}^1|0,1\rangle\langle 0,1|} + \underbrace{|1,1\rangle\langle 1,1|\,\overline{\vartheta}^1\overline{\vartheta}^2}_{-\overline{\vartheta}^2\overline{\vartheta}^1|1,1\rangle\langle 1,1|}\Big]$$

$$= -\underbrace{\int d\overline{\vartheta}^2 d\overline{\vartheta}^1\;\overline{\vartheta}^2\overline{\vartheta}^1}_{(-i)^2}\;\mathbb{1}_{\mathcal{H}_a} = \mathbb{1}_{\mathcal{H}_a}\quad , \tag{A.15}$$

that is

$$\boxed{-i\int d^2\overline{\boldsymbol{\vartheta}}\;\;|\overline{\boldsymbol{\vartheta}}\rangle\langle\overline{\boldsymbol{\vartheta}}| = \mathbb{1}_{\mathcal{H}_a}.} \tag{A.16}$$

Similarly one shows that

$$\boxed{i\int d^2\boldsymbol{\vartheta}\;\;|\boldsymbol{\vartheta}\rangle\langle\boldsymbol{\vartheta}| = \mathbb{1}_{\mathcal{H}_a}.} \tag{A.17}$$

For more information on coherent states and their integrals see Ohnuki[16], Glauber[17] and Klauder[18].

A.5 Reversing Wick-rotations

Wick-rotations generally involve coordinate and field transformations of the kind $\widetilde{x}^\mu := \beta^\mu x^\mu$, $\widetilde{A}_\mu := \alpha_\mu A_\mu$ with $\alpha_\mu, \beta_\mu \in \mathbb{C}\setminus\{0\}$. Set $B := \mathrm{diag}(\beta^1,..,\beta^D)$ and $\beta := \det B$. Suppose the effective action $\Gamma_E[\widetilde{\mathbf{A}}]$ is given as a function of the modified field $\widetilde{\mathbf{A}}$, which is its self a function of $\widetilde{\mathbf{x}}$. Then Γ_E can also be interpreted as a functional of $\mathbf{A}$, by means of $\Gamma_M[\mathbf{A}] := \Gamma_E[\widetilde{\mathbf{A}}(\mathbf{A})]$. Similarly, $\widetilde{\mathbf{A}}$ can be interpreted also as a field of $\mathbf{x}$ by means of $\widetilde{\mathbf{A}} = \widetilde{\mathbf{A}}(\widetilde{\mathbf{x}}(\mathbf{x}))$, although care has to be taken when creating N-point functions from Γ.

For once, one has

$$\frac{\delta^N\Gamma_M}{\delta A^{\mu_1}(\widetilde{\mathbf{x}}_1)\ldots\delta A^{\mu_N}(\widetilde{\mathbf{x}}_N)} = \underbrace{\frac{\delta^N\Gamma_E}{\delta\widetilde{A}^{\mu_1}(\widetilde{\mathbf{x}}_1)\ldots\delta\widetilde{A}^{\mu_N}(\widetilde{\mathbf{x}}_N)}}_{=:\Gamma_E^{\mu_1..\mu_N}[\widetilde{\mathbf{x}}_1,..,\widetilde{\mathbf{x}}_N]}\cdot\alpha_{\mu_1}\cdot..\cdot\alpha_{\mu_N} \tag{A.18}$$

On the other hand, N-point correlator functions are to be used as (tensor valued) measure densities over $\widetilde{\mathbf{x}}$ or $\mathbf{x}$ for that matter, which implies the transformation

$$\underbrace{\frac{\delta^N\Gamma_M}{\delta A^{\mu_1}(\mathbf{x}_1)\ldots\delta A^{\mu_N}(\mathbf{x}_N)}}_{=:\Gamma_M^{\mu_1..\mu_N}[\mathbf{x}_1,..,\mathbf{x}_N]} = \frac{\delta^N\Gamma_M}{\delta A^{\mu_1}(\widetilde{\mathbf{x}}_1)\ldots\delta A^{\mu_N}(\widetilde{\mathbf{x}}_N)}\cdot\underbrace{\det{}^N\left[\frac{d\widetilde{\mathbf{x}}}{d\mathbf{x}}\right]}_{\beta^N}, \tag{A.19}$$

so that

$$\boxed{\Gamma_M^{\mu_1..\mu_N}[\mathbf{x}_1,..,\mathbf{x}_N] = \beta^N\alpha_{\mu_1}..\alpha_{\mu_N}\cdot\Gamma_E^{\mu_1..\mu_N}[\widetilde{\mathbf{x}}_1,..,\widetilde{\mathbf{x}}_N].} \tag{A.20}$$

Furthermore, when Fourier-transforming additional care has to be taken, since

$$\mathcal{F}\left\{\Gamma_M^{\mu_1..\mu_N}\right\}(\mathbf{p}^1,..,\mathbf{p}^N) := \frac{1}{(2\pi)^{\frac{DN}{2}}}\int \underbrace{d\mathbf{x}_1..d\mathbf{x}_N}_{\beta^{-N}d\widetilde{\mathbf{x}}_1..d\widetilde{\mathbf{x}}_N}\ \Gamma_M^{\mu_1..\mu_N}[\mathbf{x}_1,..,\mathbf{x}_N]\ \exp\left[i\sum_{k=1}^{N}\underbrace{\mathbf{p}^k\mathbf{x}_k}_{\mathbf{p}^k B^{-1}\widetilde{\mathbf{x}}_k}\right]$$

$$\overset{(A.20)}{=} \frac{\alpha_{\mu_1}..\alpha_{\mu_N}}{(2\pi)^{\frac{DN}{2}}}\int d\widetilde{\mathbf{x}}_1..d\widetilde{\mathbf{x}}_N\ \Gamma_E^{\mu_1..\mu_N}[\widetilde{\mathbf{x}}_1,..,\widetilde{\mathbf{x}}_N]\exp\left[i\sum_{k=1}^{N}\mathbf{p}^k B^{-1}\widetilde{\mathbf{x}}_k\right]$$

$$= \alpha_{\mu_1}\cdot..\cdot\alpha_{\mu_N}\cdot\mathcal{F}\left\{\Gamma_E^{\mu_1..\mu_N}\right\}\left[B^{-1}\mathbf{p}_1^1,..,B^{-1}\mathbf{p}^N\right]. \tag{A.21}$$

Example: In the case of a Wick contraction of the type $(\alpha_1,..,\alpha_D) = (-i,1,..,1)$, $(\beta_1,..,\beta_D) = (i,1,..,1)$ and a 2-point function

$$\Gamma_E^{\mu\nu}[\mathbf{p},\mathbf{q}] = f(\mathbf{pq})\cdot\delta(\mathbf{p}+\mathbf{q})\cdot[p_\mu q_\nu - (\mathbf{pq})\cdot\delta_{\mu\nu}] \quad , \quad C:\text{const} \quad , \tag{A.22}$$

one obtains

$$\Gamma_M^{\mu\nu}[\mathbf{p},\mathbf{q}] = f(\mathbf{p}B^{-2}\mathbf{q})\cdot\underbrace{\delta\left[B^{-1}(\mathbf{p}+\mathbf{q})\right]}_{\det B\cdot\delta(\mathbf{p}+\mathbf{q})}\cdot\alpha_\mu\alpha_\nu\cdot\left[\frac{p_\mu q_\nu}{\beta_\mu\beta_\nu} - (\mathbf{p}B^{-2}\mathbf{q})\cdot\delta_{\mu\nu}\right]$$

$$= if(-g(\mathbf{p},\mathbf{q}))\cdot\delta(\mathbf{p}+\mathbf{q})\cdot[p^\mu q^\nu - g(\mathbf{p},\mathbf{q})\cdot g_{\mu\nu}] \quad . \tag{A.23}$$

A.6 The polarization tensor

Consider the Dirac theory of QED, with Lagrangian

$$\mathcal{L} = \mathcal{L}_{\text{sp}} + \mathcal{L}_{\text{em}} + \mathcal{L}_{\text{int}}, \tag{A.24}$$

whereas

$$\mathcal{L}_{\text{sp}} = i\overline{\psi}\gamma^\mu\partial_\mu\psi - m\overline{\psi}\psi \quad , \quad \mathcal{L}_{\text{em}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad , \quad \mathcal{L}_{\text{int}} = e\overline{\psi}\gamma^\mu A_\mu\psi \tag{A.25}$$

quantized with normal-ordering description

$$\hat{\mathcal{H}}_{\text{int}} = -:e\hat{\overline{\psi}}\gamma^\mu\hat{\psi}\hat{A}_\mu: . \tag{A.26}$$

Let

$$S_F^{\mu\nu}(\mathbf{x}_2-\mathbf{x}_1) := -i\underbrace{\hat{\psi}^\mu(\mathbf{x}_2)\hat{\overline{\psi}}^\nu}(\mathbf{x}_1)$$

$$D_F^{\mu\nu}(\mathbf{x}_2-\mathbf{x}_1) := -i\underbrace{\hat{A}^\mu(\mathbf{x}_2)\hat{A}^\nu}(\mathbf{x}_1) \tag{A.27}$$

be the (free) spinor and photon propagators respectively.

The 2nd order contribution to the S-matrix of the interacting fields, takes the form[13]

$$\hat{S}^{(2)} = \frac{(ie)^2}{2!}\int d\mathbf{x}_1\ d\mathbf{x}_2\ T\left[:\hat{\overline{\psi}}(\mathbf{x}_1)\gamma^\mu\hat{\psi}(\mathbf{x}_1)\hat{A}_\mu(\mathbf{x}_1):\ :\hat{\overline{\psi}}(\mathbf{x}_2)\gamma^\mu\hat{\psi}(\mathbf{x}_2)\hat{A}_\nu(\mathbf{x}_2):\right]$$

$$= \cdots + \underbrace{\frac{(ie)^2}{2!}\int d\mathbf{x}_1\ d\mathbf{x}_2\ :\hat{\overline{\psi}}(\mathbf{x}_1)\gamma^\mu\hat{\psi}(\mathbf{x}_1)\hat{\overline{\psi}}(\mathbf{x}_2)\gamma^\nu\hat{\psi}(\mathbf{x}_2)\hat{A}_\mu(\mathbf{x}_1)\hat{A}_\nu(\mathbf{x}_2):}_{\text{vacuum polarization term}}, \tag{A.28}$$

with the so called *polarization tensor* $\Pi^{\mu\nu}$, defined as[36]

$$\begin{aligned}\frac{i}{4\pi}\Pi^{\mu\nu}(\mathbf{x}_1,\mathbf{x}_2) &:= (ie)^2\cdot \hat{\overline{\psi}}(\mathbf{x}_1)\gamma^\mu\hat{\psi}(\mathbf{x}_1)\hat{\overline{\psi}}(\mathbf{x}_2)\gamma^\nu\hat{\psi}(\mathbf{x}_2)\\ &= -(ie)^2\,\mathrm{tr}\left[iS_F(\mathbf{x}_2-\mathbf{x}_1)\gamma^\mu iS_F(\mathbf{x}_1-\mathbf{x}_2)\gamma^\nu\right]\\ &=: R^{\mu\nu}(\mathbf{x}_2-\mathbf{x}_1).\end{aligned}\tag{A.29}$$

We consider the vacuum polarization contribution to the S matrix element for photon-photon scattering

$$\begin{aligned}iD^{\mathrm{int}}_{F,\varkappa\lambda}(\mathbf{x},\mathbf{y}) &:= \langle 0|\,T\left[\hat{A}_\varkappa(\mathbf{x})\hat{A}_\lambda(\mathbf{y})S\right]|0\rangle\\ &= \underbrace{\langle 0|\,T\left[\hat{A}_\varkappa(\mathbf{x})\hat{A}_\lambda(\mathbf{y})\right]|0\rangle}_{iD_{F,\varkappa\lambda}(\mathbf{x}-\mathbf{y})} + \underbrace{[\text{1st order terms}]}_{0\ (\text{see [13]})} + \langle 0|\,T\left[\hat{A}_\varkappa(\mathbf{x})\hat{A}_\lambda(\mathbf{y})S^{(2)}\right]|0\rangle\\ &= iD_{F,\varkappa\lambda}(\mathbf{x}-\mathbf{y}) + (\dots)\\ &\quad+\frac{1}{2!}\int d\mathbf{x}_1\ d\mathbf{x}_2\ \hat{A}_\varkappa(\mathbf{x})\hat{A}_\mu(\mathbf{x}_1)\hat{A}_\lambda(\mathbf{y})\hat{A}_\nu(\mathbf{x}_2)\cdot R^{\mu\nu}(\mathbf{x}_2-\mathbf{x}_1)\\ &\quad+\frac{1}{2!}\int d\mathbf{x}_1\ d\mathbf{x}_2\ \hat{A}_\varkappa(\mathbf{x})\hat{A}_\nu(\mathbf{x}_2)\hat{A}_\lambda(\mathbf{y})\hat{A}_\mu(\mathbf{x}_1)\cdot R^{\mu\nu}(\mathbf{x}_2-\mathbf{x}_1)\\ &= iD_{F,\varkappa\lambda}(\mathbf{x}-\mathbf{y}) + (\dots) + \underbrace{\int d\mathbf{x}_1\ d\mathbf{x}_2\ iD_{F,\varkappa\mu}(\mathbf{x}-\mathbf{x}_1)R^{\mu\nu}(\mathbf{x}_1-\mathbf{x}_2)iD_{F,\nu\lambda}(\mathbf{x}_2-\mathbf{y})}_{[iD_{F,\varkappa\mu}*R^{\mu\nu}*iD_{F,\nu\lambda}](\mathbf{x}-\mathbf{y})=:iD^{\mathrm{pol}}_{F,\varkappa\lambda}(\mathbf{x}-\mathbf{y})}\end{aligned}\tag{A.30}$$

By the convolution theorem A.9, the vacuum-polarization contribution has the Fourier-transform

$$i\widetilde{D}^{\mathrm{pol}}_{F,\varkappa\lambda}(\mathbf{p}) = (2\pi)^D\cdot i\widetilde{D}_{F,\varkappa\mu}(\mathbf{p})\cdot\frac{i\widetilde{\Pi}^{\mu\nu}(\mathbf{p})}{4\pi}\cdot i\widetilde{D}_{F,\nu\lambda}(\mathbf{p})\quad .\tag{A.31}$$

Recall that the spinor propagator has the Fourier-transform

$$\widetilde{S}_F(\mathbf{p}) = \frac{1}{(2\pi)^{\frac{D}{2}}}\frac{i}{\gamma^\mu p_\mu - m + i\varepsilon}.\tag{A.32}$$

Using appendix A.8, we can thus write the Fourier-transform of (A.29) as

$$\frac{i\widetilde{\Pi}^{\mu\nu}(\mathbf{p})}{4\pi} = \frac{(ie)^2}{(2\pi)^{\frac{D}{2}}}\int\frac{d\mathbf{q}}{(2\pi)^D}\,\mathrm{tr}\left[\frac{1}{\gamma^\lambda q_\lambda - m + i\varepsilon}\gamma^\mu\frac{1}{\gamma^\lambda(q_\lambda - p_\lambda) - m + i\varepsilon}\gamma^\nu\right]\quad .\tag{A.33}$$

A.7 On Fourier-Transforms of 2-point functions

Consider a function $f = f(\mathbf{x},\mathbf{y})$, depending actually only on the difference $(\mathbf{x}-\mathbf{y})$, that is, $f(\mathbf{x},\mathbf{y}) = g(\mathbf{x}-\mathbf{y})$ for some function g. Taking the Fourier-Transform yields

$$\begin{aligned}\mathcal{F}\left\{f\right\}(\mathbf{p},\mathbf{q}) &= \frac{1}{(2\pi)^D}\int d^D\mathbf{x}\ d^D\mathbf{y}\ e^{i\mathbf{p}\mathbf{x}+i\mathbf{q}\mathbf{y}}\underbrace{f(\mathbf{x},\mathbf{y})}_{g(\mathbf{x}-\mathbf{y})} \overset{\mathbf{z}:=\mathbf{x}-\mathbf{y}}{=} \frac{1}{(2\pi)^D}\underbrace{\int d\mathbf{y}\ e^{i(\mathbf{p}+\mathbf{q})\mathbf{y}}}_{(2\pi)^D\delta(\mathbf{p}+\mathbf{q})}\int d\mathbf{z}\ e^{i\mathbf{p}\mathbf{z}}\ g(\mathbf{z})\\ &= (2\pi)^{\frac{D}{2}}\cdot\delta(\mathbf{p}+\mathbf{q})\cdot\mathcal{F}\left\{g\right\}(\mathbf{p})\end{aligned}\tag{A.34}$$

[36]Note that $\Pi^{\mu\nu}(\mathbf{x})$ is symmetric in μ,ν and even in $\mathbf{x}$.

A.8 On Fourier-transforms of traces

Let $\mathbb{A}, \mathbb{B}$ be $n \times n$-matrix valued fields and consider the Fourier-transform of the field $A(\mathbf{x})B(\mathbf{x})$. Then by the convolution theorem A.9:

$$\mathcal{F}\left\{\mathrm{tr}(\mathbb{A}\mathbb{B})\right\}(\mathbf{p}) = \sum_{k=1}^{n} \mathcal{F}\left\{A^{kl}B^{lk}\right\}(\mathbf{p}) = \frac{1}{(2\pi)^{\frac{D}{2}}} \sum_{k=1}^{n} \left[\mathcal{F}(A^{kl}) * \mathcal{F}(B^{lk})\right](\mathbf{p})$$

$$= \int \frac{d\mathbf{q}}{(2\pi)^{\frac{D}{2}}} \,\mathrm{tr}\left[\mathcal{F}(\mathbb{A})(\mathbf{p}-\mathbf{q}) \cdot \mathcal{F}(\mathbb{B})(\mathbf{q})\right] \quad . \tag{A.35}$$

A.9 The convolution theorem

Let f, g be two Schwartz functions on $\mathbb{R}^D$. Then[37]:

$$\mathcal{F}(fg) = (2\pi)^{-\frac{D}{2}} \mathcal{F}(f) * \mathcal{F}(g) \quad , \quad \mathcal{F}(f * g) = (2\pi)^{\frac{D}{2}} \mathcal{F}(f) \cdot \mathcal{F}(g). \tag{A.36}$$

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